

# The Second Main Theorem for Holomorphic Curves into Semi-Abelian Varieties II \*

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## Abstract

We establish the second main theorem with the best truncation level one

$$T(r; \omega_{\bar{Z}, J_k(f)}) \leq N_1(r; J_k(f)^* Z) + \epsilon T_f(r) ||_\epsilon$$

for the  $k$ -jet lift  $J_k(f) : \mathbf{C} \rightarrow J_k(A)$  of an algebraically non-degenerate entire holomorphic curve  $f : \mathbf{C} \rightarrow A$  into a semi-abelian variety  $A$  and an arbitrary algebraic reduced subvariety  $Z$  of  $J_k(A)$ ; the low truncation level is important for applications. Finally we give some applications, including the solution of a problem posed by Mark Green (1974).

## 1 Introduction and main result

Let  $f : \mathbf{C} \rightarrow V$  be a holomorphic curve into a complex projective manifold  $V$  with Zariski dense image and let  $D$  be an effective reduced divisor on  $V$ . Under some ampleness condition for the space  $H^0(V, \Omega_V^1(\log D))$  of logarithmic 1-forms along  $D$  we proved in [N77], [N81] the following inequalities of the second main theorem type,

$$\begin{aligned} \kappa T_f(r) &\leq N(r; f^* D) + O(\log r) + O(\log T_f(r)), \\ \kappa' T_f(r) &\leq N_1(r; f^* D) + O(\log r) + O(\log T_f(r)), \end{aligned}$$

where  $T_f(r)$  denotes the order function of  $f$ ,  $N(r; f^* D)$  (resp.  $N_l(r; f^* D)$ ) the counting function (resp. truncated to level  $l$ ) of the pull-backed divisor  $f^* D$ , and  $\kappa$  and  $\kappa'$  are positive constants (cf. §2). It is an interesting and fundamental problem to determine the constant  $\kappa$  or  $\kappa'$ . In the case where  $V$  is the compactification of a semi-abelian variety  $A$  this problem is related to what kind of compactification  $V$  of  $A$  we take. In our former

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paper [NWY02] we proved that for a holomorphic curve  $f : \mathbf{C} \rightarrow A$  into a semi-abelian variety  $A$  and an algebraic divisor  $D$  on  $A$ ,

$$(1.1) \quad T_f(r; L(\bar{D})) \leq N_l(r; f^*D) + O(\log r) + O(\log T_f(r; L(\bar{D})))||.$$

Here we used a compactification  $\bar{A}$  of  $A$  such that the maximal affine subgroup  $(\mathbf{C}^*)^t$  of  $A$  was compactified by  $(\mathbf{P}^1(\mathbf{C}))^t$ , and we assumed a boundary condition (Condition 4.11 in [NWY02]) for the closure  $\bar{D}$  of  $D$  in  $\bar{A}$ ; this roughly meant the divisor  $\bar{D} + (\bar{A} \setminus A)$  to be in general position and has been expected to be removed by a suitable choice of a compactification of  $A$ . It is an important and very interesting problem to take the truncation level  $l$  as small as possible.

Let  $X_k(f)$  denote the Zariski closure of the image of the  $k$ -jet lift of  $f$  in the  $k$ -jet space  $J_k(A)$  over  $A$ . The purpose of this paper is to prove (cf. §§2, 3 for notation)

**Main Theorem.** *Let  $A$  be a semi-abelian variety. Let  $f : \mathbf{C} \rightarrow A$  be a holomorphic curve with Zariski dense image.*

(i) *Let  $Z$  be an algebraic reduced subvariety of  $X_k(f)$  ( $k \geq 0$ ). Then there exists a compactification  $\bar{X}_k(f)$  of  $X_k(f)$  such that*

$$(1.2) \quad T(r; \omega_{\bar{Z}, J_k(f)}) \leq N_1(r; J_k(f)^*Z) + \epsilon T_f(r) ||_\epsilon, \quad \forall \epsilon > 0,$$

*where  $\bar{Z}$  is the closure of  $Z$  in  $\bar{X}_k(f)$ .*

(ii) *Moreover, if  $\text{codim}_{X_k(f)} Z \geq 2$ , then*

$$(1.3) \quad T(r; \omega_{\bar{Z}, J_k(f)}) \leq \epsilon T_f(r) ||_\epsilon, \quad \forall \epsilon > 0.$$

(iii) *In the case when  $k = 0$  and  $Z$  is an effective divisor  $D$  on  $A$ , the compactification  $\bar{A}$  of  $A$  can be chosen as smooth, equivariant with respect to the  $A$ -action, and independent of  $f$ ; furthermore, (1.2) takes the form*

$$(1.4) \quad T_f(r; L(\bar{D})) \leq N_1(r; f^*D) + \epsilon T_f(r; L(\bar{D})) ||_\epsilon, \quad \forall \epsilon > 0.$$

Note that in the above estimate (1.2), (1.3) or (1.4) the small error term “ $\epsilon T_f(r)$ ” cannot be replaced by “ $O(\log r) + O(\log T_f(r))$ ” (see [NWY02] Example (5.36)).

The Main Theorem is an advancement of [NWY02] and [Y04]. When  $A$  is an abelian variety, (1.4) was proved by Yamanoi [Y04] (cf. [Y04] (3.1.8)). A key of the proof of (1.1) in [NWY02] was Lemma 5.6 at p. 147, and here we will again use the same idea for jets of jets (see Claim 4.13).

There is a related result due to Siu-Yeung [SY03], where they obtained (1.1) with an improved truncation level  $l = l(D)$  dependent only on the Chern numbers of  $D$ . In their proof the key was Claim 1 at p. 443 which was the same as [NYW02] Lemma 5.6 restricted to the abelian case with a computation of intersection numbers.

It is interesting to observe that the error term being “ $O(\log r) + O(\log T_f(r; L(\bar{D})))||$ ”, the truncation level  $l$  in (1.1) has to depend on  $D$ , but the error term being allowed to be a little bit large, “ $\epsilon T_f(r; L(\bar{D}))||_\epsilon$ ”,  $l$  can be one, the smallest possible. In applications, the truncation of level one is very definite.

To deal with semi-abelian varieties the main difficulties are caused by the following two points:

- (i) Semi-abelian varieties are not compact and need some good compactifications.
- (ii) There is no Poincaré reducibility theorem for semi-abelian varieties.

It is also noted that a part of the proof of the Main Theorem for abelian varieties in [Y04] does not hold for semi-abelian varieties ([Y04] §3 Claim), and that a different and considerably simpler proof for that part will be provided (see Lemma 6.1).

In §7 we will give two applications of the Main Theorem. The first is a complete affirmative answer to a conjecture of M. Green [G74] pp. 229–230 (cf. Theorem 7.2). The second is a non-existence theorem for some differential equations defined over semi-abelian varieties (cf. Theorem 7.6).

More applications will be obtained in [NWY05].

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## 2 Notation

The notation here follows that of [NWY02]. For a general reference of this section, cf. [NO<sup>84</sup><sub>90</sub>]. For convenience we recall some of definitions. Let  $M$  be a compact complex manifold and let  $\omega$  be a smooth (1,1)-form on  $M$ . Let  $f : \mathbf{C} \rightarrow M$  be a holomorphic curve into  $M$ . We define the order function of  $f$  with respect to  $\omega$  by

$$(2.1) \quad T_f(r; \omega) = \int_1^r \frac{dt}{t} \int_{|z| < t} f^* \omega \quad (r > 1).$$

If  $M$  is Kähler and  $d\omega = 0$ ,

$$T_f(r; \omega) = T_f(r; \omega') + O(1)$$

for a  $d$ -closed  $(1,1)$ -form  $\omega'$  in the same cohomology class  $[\omega] \in H^2(M, \mathbf{R})$ . Therefore we set, up to  $O(1)$ -term,

$$(2.2) \quad T_f(r; [\omega]) = T_f(r; \omega).$$

Let  $L \rightarrow M$  be a hermitian line bundle with Chern class  $c_1(L)$ . Then we set

$$T_f(r; L) = T_f(r; c_1(L)),$$

which is defined again up to  $O(1)$ -term.

For a divisor  $D$  on  $M$  we denote by  $L(D)$  the line bundle determined by  $D$ .

Let  $E = \sum_{\mu=1}^{\infty} \nu_{\mu} z_{\mu}$  be a divisor on  $\mathbf{C}$  with distinct  $z_{\mu} \in \mathbf{C}$ . Then we set

$$\text{ord}_z E = \begin{cases} \nu_{\mu}, & z = z_{\mu}, \\ 0, & z \notin \{z_{\mu}\}. \end{cases}$$

We define the counting functions of  $E$  truncated to  $l \leq \infty$  by

$$n_l(t; E) = \sum_{\{|z_{\mu}| < t\}} \min\{\nu_{\mu}, l\},$$

$$N_l(r; E) = \int_1^r \frac{n_l(t; E)}{t} dt.$$

We define the counting functions of  $E$  by

$$n(t; E) = n_{\infty}(t; E), \quad N(r; E) = N_{\infty}(r; E).$$

*Definition of small terms.* (i) For a line bundle  $L \rightarrow M$  and a holomorphic curve  $f : \mathbf{C} \rightarrow M$  we denote by  $S_f(r; L)$  such a small term as

$$S_f(r; L) = O(\log r) + O(\log^+ T_f(r; L)),$$

where “ $||$ ” stands for the inequality to hold for every  $r > 1$  outside a Borel set of finite Lebesgue measure.

(ii) Let  $h(r)$  ( $r > 1$ ) be a real valued function. We write

$$h(r) \leq \epsilon T_f(r; L) ||_{\epsilon}, \quad \forall \epsilon > 0,$$

if the stated inequality holds for every  $r > 1$  outside a Borel set of finite Lebesgue measure, dependent on an arbitrarily given  $\epsilon > 0$ .

*Definition.* When  $M$  is an algebraic variety, we say that  $f : \mathbf{C} \rightarrow M$  is *algebraically* (resp. *non-*) *degenerate* if the image  $f(\mathbf{C})$  is (resp. not) contained in a proper algebraic subset of  $M$ .

The following follows from general properties of order functions ([NO<sub>84</sub><sup>84</sup>/<sub>90</sub>]).

**Lemma 2.3** *Let  $f : \mathbf{C} \rightarrow M$  be a holomorphic curve into a complex projective manifold  $M$  and  $H$  a line bundle on  $M$ . Assume that  $H$  is big, and that  $f$  is algebraically non-degenerate. Then*

$$T_f(r, L) = O(T_f(r, H))$$

*for every line bundle  $L$  on  $M$ .*

If  $f : \mathbf{C} \rightarrow M$  is algebraically degenerate, we may consider the Zariski closure  $N$  of  $f(\mathbf{C})$  and a desingularization  $\tau : \tilde{N} \rightarrow N$ . Then  $f$  lifts to a map to  $\tilde{N}$  and  $\tau^*(H|_N)$  is big on  $\tilde{N}$  for every ample line bundle  $H$  on  $M$ . As a consequence we obtain:

**Lemma 2.4** *Let  $f : \mathbf{C} \rightarrow M$  be a holomorphic curve into a complex projective manifold  $M$ . Let  $h(r)$  be a non-negative valued function in  $r > 1$ . Then  $h(r) = S_f(r; H)$  holds for every ample line bundle if and only if it holds for at least one ample line bundle.*

*Similarly the statement  $h(r) \leq \epsilon T_f(r; H) + o(r)$ ,  $\forall \epsilon > 0$ , respectively  $h(r) = O(T_f(r; H))$  holds for every ample line bundle  $H$  if and only if it holds for at least one ample line bundle.*

If one of these conditions holds for one and therefore for all ample line bundles  $H$ , we simply write  $h(r) = S_f(r)$  (resp.  $h(r) \leq \epsilon T_f(r) + o(r)$ ,  $h(r) = O(T_f(r))$ ).

For a quasi-projective manifold  $V$  and for a holomorphic curve  $f : \mathbf{C} \rightarrow V$  we write simply  $T_f(r) = T_f(r; H)$  for the order function with respect to an ample line bundle  $H$  over a projective compactification  $\bar{M}$  of  $M$  if the choice of  $\bar{M}$  and  $H$  do not matter.

The following related property of order functions will be frequently used ([NO<sub>90</sub> Lemma (6.1.5)).

**Lemma 2.5** *Let  $\eta : V \rightarrow W$  be a rational mapping between quasi-projective manifolds  $V$  and  $W$ . Then for an algebraically non-degenerate holomorphic curve  $f : \mathbf{C} \rightarrow V$*

$$T_{\eta \circ f}(r) = O(T_f(r)).$$

*Moreover, if  $\eta$  is generically finite, then*

$$T_f(r) = O(T_{\eta \circ f}(r)).$$

We define the proximity function  $m_f(r; \mathcal{I})$  not only for divisors but also for a coherent ideal sheaf  $\mathcal{I}$  of the structure sheaf  $\mathcal{O}_M$  over  $M$ . Let  $\{U_j\}$  be a finite open covering of  $M$  such that

- (i) there is a partition of unity  $\{c_j\}$  associated with  $\{U_j\}$ ,

- (ii) there are finitely many sections  $\sigma_{jk} \in \Gamma(U_j, \mathcal{I})$ ,  $k = 1, 2, \dots$ , generating every fiber  $\mathcal{I}_x$  over  $x \in U_j$ .

Setting  $\rho_{\mathcal{I}}(x) = \left( \sum_j c_j(x) \sum_k |\sigma_{jk}(x)|^2 \right)^{1/2}$ , we take a positive constant  $C$  so that

$$C\rho_{\mathcal{I}}(x) \leq 1, \quad x \in M.$$

Using the compactness of  $M$ , one easily verifies that, up to addition by a bounded continuous function on  $M$ ,  $\log \rho_{\mathcal{I}}$  is independent of the choices of the open covering, the partition of unity, the local generators of the ideal sheaf  $\mathcal{I}$ , and the constant  $C$ .

We define the proximity function of  $f$  for  $\mathcal{I}$  or for the subspace (may be non-reduced)  $Y = (\text{Supp } \mathcal{O}_M/\mathcal{I}, \mathcal{O}/\mathcal{I})$  by

$$(2.6) \quad m_f(r; Y) = m_f(r; \mathcal{I}) = \int_{|z|=r} \log \frac{1}{C\rho_{\mathcal{I}}(f(re^{i\theta}))} \frac{d\theta}{2\pi} \quad (\geq 0),$$

provided that  $f(\mathbf{C}) \not\subset \text{Supp } Y$ . Note that if  $\mathcal{I}$  is the ideal sheaf defined by an effective divisor  $D$  on  $M$ ,  $m_f(r; \mathcal{I})$  coincides  $m_f(r; D)$  defined in [NWY02] up to  $O(1)$ -term. The function  $\rho_{\mathcal{I}} \circ f(z)$  is smooth over  $\mathbf{C} \setminus f^{-1}(\text{Supp } Y)$ . For  $z_0 \in f^{-1}(\text{Supp } Y)$  choose an open neighborhood  $U$  of  $z_0$  and a positive integer  $\nu$  such that  $f^*\mathcal{I} = ((z - z_0)^\nu)$ . Then

$$\log \rho_{\mathcal{I}} \circ f(z) = \nu \log |z - z_0| + \psi(z), \quad z \in U.$$

for some smooth function  $\psi(z)$  defined on  $U$ . We define the counting function  $N(r; f^*\mathcal{I})$  and  $N_l(r; f^*\mathcal{I})$  by using  $\nu$  in the same way as using  $\text{ord}_{z_0}(E)$  in the definition of  $N(r; E)$  and  $N_l(r; E)$ . Moreover we define

$$(2.7) \quad \begin{aligned} \omega_{\mathcal{I}, f} &= \omega_{Y, f} = -dd^c \psi(z) = -\frac{i}{2\pi} \partial \bar{\partial} \psi(z) \\ &= dd^c \log \frac{1}{\rho_{\mathcal{I}} \circ f(z)} \quad (z \in U), \end{aligned}$$

which is well-defined on  $\mathbf{C}$  as a smooth  $(1,1)$ -form. The order function of  $f$  for  $\mathcal{I}$  or  $Y$  is defined by

$$(2.8) \quad T(r; \omega_{\mathcal{I}, f}) = T(r; \omega_{Y, f}) = \int_1^r \frac{dt}{t} \int_{|z| < t} \omega_{\mathcal{I}, f}.$$

When  $\mathcal{I}$  defines a divisor  $D$  on  $M$ , we see that

$$T(r; \omega_{\mathcal{I}, f}) = T_f(r; L(D)) + O(1).$$

Let  $\mathcal{I}_i$  ( $i = 1, 2$ ) be coherent ideal sheaves of  $\mathcal{O}_M$  and let  $Y_i$  be the subspace defined by  $\mathcal{I}_i$ . We write  $Y_1 \supset Y_2$  if  $\mathcal{I}_1 \subset \mathcal{I}_2$ .

**Theorem 2.9** *Let  $f : \mathbf{C} \rightarrow M$  and  $\mathcal{I}$  be as above. Then we have the following:*

(i) (First Main Theorem)

$$T(r; \omega_{\mathcal{I}, f}) = N(r; f^* \mathcal{I}) + m_f(r; \mathcal{I}) - m_f(1; \mathcal{I}).$$

(ii) *If  $M$  is projective,  $m_f(r; \mathcal{I}) = O(T_f(r))$ .*

(iii) *Let  $\mathcal{I}_i$  ( $i = 1, 2$ ) be coherent ideal sheaves of  $\mathcal{O}_M$  and let  $Y_i$  be the subspace defined by  $\mathcal{I}_i$ . If  $\mathcal{I}_1 \subset \mathcal{I}_2$  or equivalently  $Y_1 \supset Y_2$ , then*

$$m_f(r; \mathcal{I}_2) \leq m_f(r; \mathcal{I}_1) + O(1),$$

*or equivalently,*

$$m_f(r; Y_2) \leq m_f(r; Y_1) + O(1).$$

(iv) *Let  $\phi : M_1 \rightarrow M_2$  be a holomorphic mappings between compact complex manifolds. Let  $\mathcal{I}_2 \subset \mathcal{O}_{M_2}$  be a coherent ideal sheaf and let  $\mathcal{I}_1 \subset \mathcal{O}_{M_1}$  be the coherent ideal sheaf generated by  $\phi^* \mathcal{I}_2$ . Then*

$$m_f(r; \mathcal{I}_1) = m_{\phi \circ f}(r; \mathcal{I}_2) + O(1).$$

(v) *Let  $\mathcal{I}_i$ ,  $i = 1, 2$  be two coherent ideal sheaves of  $\mathcal{O}_M$ . Suppose that  $f(\mathbf{C}) \not\subset \text{Supp}(\mathcal{O}_M / \mathcal{I}_1 \otimes \mathcal{I}_2)$ . Then we have*

$$T(r; \omega_{\mathcal{I}_1 \otimes \mathcal{I}_2, f}) = T(r; \omega_{\mathcal{I}_1, f}) + T(r; \omega_{\mathcal{I}_2, f}) + O(1).$$

*Proof.* (i) This immediately follows from the well-known Jensen formula (cf. [NO<sub>90</sub><sup>84</sup>] Theorem (5.2.15)).

(ii) Let  $Y$  be the subvariety defined by  $\mathcal{I}$ . There is an ample divisor  $D$  on  $M$  such that  $D \supset Y$  (counting multiplicities). It follows from Theorem (2.9) (iii) that

$$m_f(r; Y) \leq m_f(r; D) \leq T_f(r; L(D)) = O(T_f(r)).$$

(iii) (iv) (v) These are immediate by definition. *Q.E.D.*

### 3 General position

*Convention 3.1* Unless explicitly stated otherwise, all varieties, morphisms, group actions, compactifications, divisors etc. are assumed to be algebraic.

### 3.1 General position

Let  $A$  be a semi-abelian variety and let  $X$  be a complex algebraic variety (possibly singular) on which  $A$  acts:

$$(a, x) \in A \times X \rightarrow a \cdot x \in X.$$

Let  $Y$  be a subvariety embedded into a Zariski open subset of  $X$ .

*Definition 3.2* We say that  $Y$  is *generally positioned in  $X$*  if the closure  $\bar{Y}$  of  $Y$  in  $X$  contains no  $A$ -orbit. If the support of a divisor  $E$  on a Zariski open subset of  $X$  is generally positioned in  $X$ , then  $E$  is said to be generally positioned in  $X$ .

Let  $\pi : X_1 \rightarrow X$  be a blow-up of smooth projective manifolds on which  $A$  acts. Let  $D$  be a divisor on  $X$  and let  $D_1$  be its strict transform. Then  $D_1 \sim \pi^*D - E$ , where  $E$  is an effective divisor with support contained in the exceptional locus of the blow-up. If  $\pi$  is the blow-up along a smooth connected submanifold  $C \subset X$ , then  $E$  is empty unless  $C \subset D$ .

**Lemma 3.3** *Assume that  $D$  is generally positioned in  $X$ . Let  $\pi : X_1 \rightarrow X$  be an equivariant blow-up. Then  $D_1 = \pi^*D$ , i.e.,  $E$  is empty.*

*Proof.* Since the blow-up is assumed to be equivariant, its center  $C$  must be an invariant subset, i.e.,  $C$  is a union of  $A$ -orbits. Now  $D$  is assumed to be generally positioned in  $X$ . This implies that  $D$  contains no  $A$ -orbit. Therefore no irreducible component of  $C$  is contained in  $D$ . *Q.E.D.*

**Corollary 3.4** *Assume that  $D$  is big and generally positioned in  $X$ . Then  $D_1$  is big, too.*

*Proof.* This is immediate from  $D_1 = \pi^*D$ . *Q.E.D.*

Unfortunately the assumption of being generally positioned can not be dropped. For example, let us consider  $X = \mathbf{P}^2(\mathbf{C})$ . Let  $D$  be a line and let  $X_1 \rightarrow X$  be the blow-up of a point  $p$  on the line  $D$ . Then  $X_1$  is a ruled surface. It admits a fibration  $\tau : X_1 \rightarrow \mathbf{P}^1(\mathbf{C})$  which arises as follows: We may identify  $\mathbf{P}^1(\mathbf{C})$  with  $\mathbf{P}(T_p\mathbf{P}^2(\mathbf{C}))$ . Then for  $x \in \mathbf{P}^2(\mathbf{C}) \setminus \{p\}$  we set  $\tau(x)$  to be the tangent line at  $p$  of the unique line in  $\mathbf{P}^2(\mathbf{C})$  connecting  $p$  and  $x$ . Now the strict transform  $D_1$  of  $D$  turns out to be a fiber of  $\tau$ . As a fiber of a holomorphic map, it can not be big. However,  $D$ , as an effective divisor on  $\mathbf{P}^2(\mathbf{C})$ , is big.

To give another example, consider the blow-up of  $\mathbf{P}^2(\mathbf{C})$  in two points  $p, q \in D$ . A blow-up decreases the self-intersection number of a curve by 1. Therefore the self-intersection

number of the strict transform  $D_2$  of  $D$  under this blow-up  $X_2 \rightarrow X$  is a curve with self-intersection number  $-1$ . As a consequence we have  $\dim H^0(X_2, L(nD_2)) = 1$  for all  $n \in \mathbf{N}$ .

Note that these examples are equivariant for a suitably chosen action of  $A = (\mathbf{C}^*)^2$ , but  $D$  is not generally positioned in  $\mathbf{P}^2(\mathbf{C})$ .

On the other hand, bigness can only be destroyed, not created via blow-up. This follows from the following fact:  $D_1 = \pi^*D - E$  where  $E$  is effective. Thus fixing a section  $\sigma \in H^0(X_1, E)$  we obtain an injection

$$H^0(X_1, L(nD_1)) \xrightarrow{\alpha} H^0(X_1, L(n\pi^*D)) \cong H^0(X, L(nD)) \quad (\forall n \in \mathbf{N})$$

given by mapping a section to its tensor product with  $\sigma^n$ . Therefore the Iitaka  $D$ -dimension can only decrease ([I71]).

**Lemma 3.5** *Let  $\pi : X_1 \rightarrow X$  be an equivariant blow-up, let  $D$  be a divisor on  $X$  which is generally positioned in  $X$ , and let  $D_1$  be its strict transform. Then  $D_1$  is generally positioned in  $X_1$ , too.*

*Proof.* If  $D_1$  would contain an  $A$ -orbit  $\Omega$ , we could infer that  $\pi(\Omega) \subset \pi(D_1) = D$ . Since  $\pi$  is assumed to be equivariant, this would imply that  $D$  contains an  $A$ -orbit, namely  $\pi(\Omega)$ . *Q.E.D.*

## 3.2 Stabilizer

Let  $A$  be a semi-abelian variety such that

$$(3.6) \quad 0 \rightarrow T \rightarrow A \xrightarrow{\pi} A_0 \rightarrow 0,$$

where  $T \cong (\mathbf{C}^*)^t$  and  $A_0$  is an abelian variety. Let  $D$  be a divisor on  $A$ . The stabilizer of  $D$  is defined by

$$(3.7) \quad \text{St}(D) = \{a \in A : a + D = D\}^0,$$

where  $\{\cdot\}^0$  denotes the identity component.

**Lemma 3.8** *Let  $D$  be an effective divisor on  $A$  and let  $\bar{D}$  be its closure in an equivariant compactification  $\bar{A}$  of  $A$ . Let  $L_0 \in \text{Pic}(A_0)$  and let  $E$  be an  $A$ -invariant divisor on  $\bar{A}$  such that  $L(\bar{D}) \cong L(E) \otimes \pi^*L_0$ . Assume that  $\text{St}(D)$  is contained in  $T$ . Then  $L_0$  is ample on  $A_0$ .*

*Proof.* By [NW04] Lemma 5.2 we obtain  $c_1(L_0) \geq 0$ . We may regard  $c_1(L_0)$  as a bilinear form on a vector space  $V$  which can be interpreted as the Lie algebra  $\text{Lie}(A_0)$  or the dual of cotangent bundle  $\Omega^1(A_0)^*$  over  $A_0$ . Assume that  $L_0$  is not ample. Then there is a vector  $v \in V \setminus \{0\}$  such that  $c_1(L_0)|_{\mathbf{C}v} \equiv 0$ . Choose a direct sum decomposition (orthogonal with respect to  $c_1(L_0)$ )  $V = \mathbf{C}v \oplus V'$  and let  $\omega$  be a  $(1, 1)$ -form which is positive on  $V'$ , but annihilates  $\mathbf{C}v$ . Then  $c_1(L) \wedge \omega^{g-1} = 0$  where  $g = \dim A_0 = \dim V$ . Let  $\Omega$  be a  $(1, 1)$ -form on  $\bar{A}$  which is positive along the fibers of  $\bar{A} \rightarrow A_0$  as constructed in [NW03] Lemma 5.1. Then

$$0 = \int_{\bar{A}} \Omega^s \wedge \pi^* (c_1(L_0) \wedge \omega^{g-1}) = \int_D \Omega^s \wedge \pi^* (\omega^{g-1})$$

By construction of  $\omega$  this implies that  $v$  is everywhere tangent to  $D$ . But in this case  $v \in \text{Lie}(A_0)$  is in the Lie algebra of the stabilizer  $\text{St}(D)$ . This is a contradiction. *Q.E.D.*

**Proposition 3.9** *Let  $\bar{A}$  be a smooth equivariant compactification of a semi-abelian variety  $A$ . Let  $D$  be an effective divisor on  $A$  and let  $\bar{D}$  be its closure in  $\bar{A}$ . Then the following properties hold.*

- (i)  $\bar{A} \setminus A$  is a divisor with only simple normal crossings.
- (ii) If  $\text{St}(D) = \{0\}$ , then  $\bar{D}$  is big on  $\bar{A}$ .

*Proof.* (i) This is [NW04] Lemma 3.4.

(ii) Due to [NW04] there is a line bundle  $L_0$  on  $A_0$  and an  $A$ -invariant divisor  $E$  on  $\bar{A}$  such that  $L(\bar{D}) \cong L(E) \otimes \pi^* L_0$ . By Lemma 3.8 the triviality of  $\text{St}(D)$  implies the ampleness of  $L_0$ .

Now consider the  $T$ -action. Evidently  $E$  is  $T$ -invariant. Since  $T$  acts only along the fibers of  $\pi : \bar{A} \rightarrow A_0$ , the line bundle  $\pi^* L_0$  is also  $T$ -invariant. It follows that for every  $g \in T$  the pull-back  $g^* D$  is linearly equivalent to  $D$ .<sup>1</sup> Next we define sets  $S_x$  for  $x \in A$  as follows:

$$S_x = \bigcap_{g \in T: g(x) \in D} g^* D.$$

By this definition we know that for every  $y \notin S_x$  there is a section  $\sigma$  in  $L(D)$  such that  $\sigma(x) = 0 \neq \sigma(y)$ . From the definition it follows furthermore that  $S_x$  is an algebraic subvariety of  $A$ . Using the  $A$ -invariant trivialization of the tangent bundle  $TA \cong A \times$

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<sup>1</sup>Actually  $g^* D \sim D$  holds for every  $g \in T$  and every  $T \cong (\mathbf{C}^*)^s$ -action on a projective manifold. This can be deduced from the fact that the Picard variety of a projective manifold contains no rational curves.

$\mathrm{Lie}(A)$  we can identify  $T_x(S_x)$  with a vector subspace of  $\mathrm{Lie}(A)$ . In this identification we obtain

$$T_x(S_x) = \cap_{g \in T: g(x) \in D} g^* D = \cap_{g \in T: g(x) \in D} T_{g(x)} D = \cap_{y \in \pi^{-1}(\pi(x)) \cap D} T_y(D).$$

Thus  $T_x(S_x)$  depends only on  $\pi(x)$ . Let  $F_x = \pi^{-1}(\pi(x))$ . Then all the points in  $F_x \cap S_x$  have the same tangent space. It follows that  $F_x \cap S_x$  is an orbit under a Lie subgroup of  $T$ . On the other hand,  $F_x \cap S_x$  is an algebraic subvariety. Therefore  $F_x \cap S_x$  is an orbit under an algebraic subgroup of  $T$ . A priori this subgroup may depend on the point  $x$ . However,  $T \cong (\mathbf{C}^*)^s$  contains only countably many algebraic subgroups. For this reason it follows that this algebraic subgroup must be the same for almost all points  $x \in A$ . Thus there is an algebraic subgroup  $H \subset T$  such that each connected component of  $S_x \cap F_x$  is a  $H$ -orbit for almost all  $x \in A$ . But this implies that  $D$  is invariant under  $H$ . Since  $\mathrm{St}(D) = \{0\}$ ,  $H$  is finite. Thus  $S_x \rightarrow A_0$  is generically finite for almost all  $x \in A$ . Combined with the ampleness of  $L_0$  this implies that  $D$  is big. *Q.E.D.*

**Proposition 3.10** *Let  $Z$  be a reduced subvariety of  $A$  and let  $\bar{Z}$  be its closure in a smooth equivariant compactification  $\bar{A}$  of  $A$ . If  $\mathrm{St}(Z) = \{0\}$ , then there is an equivariant blow-up  $\bar{A}^\dagger \rightarrow \bar{A}$  such that the strict transform of  $\bar{Z}$  is generally positioned in  $\bar{A}^\dagger$ .*

*In particular, there exists a smooth equivariant compactification of  $A$  in which  $Z$  is generally positioned.*

*Proof.* We can find an effective reduced divisor  $D$  on  $A$  such that  $D \supset Z$  and  $\mathrm{St}(D) = \{0\}$ . Thus it suffices to assume that  $Z = D$ , a divisor. Using a result of Vojta ([V99] Theorem 2.4 (2)) we obtain a (possibly singular) equivariant completion  $\hat{i} : A \hookrightarrow \hat{A}$  such that  $D$  is generally positioned in  $\hat{A}$ . Consider the diagonal embedding  $j : A \hookrightarrow \bar{A} \times \hat{A}$  given by  $j = (i, \hat{i})$  and let  $\bar{A}'$  denote the closure of the image  $j(A)$ . Let  $\bar{A}^\dagger \rightarrow \bar{A}'$  be an equivariant desingularization (cf. [Hi64], [BM97]). Then the composed map  $\bar{A}^\dagger \rightarrow \bar{A}$  is a blow-up of  $\bar{A}$ . Considering the natural projection  $\bar{A}^\dagger \rightarrow \hat{A}$ , we conclude, as in Lemma 3.5, that  $D$  is generally positioned in  $\bar{A}^\dagger$ . *Q.E.D.*

**Proposition 3.11** *Let  $A$  be a semi-abelian variety, let  $A \rightarrow \bar{A}$  be an equivariant compactification and let  $Z$  be a subvariety of  $A$ . Then there is an equivariant blow-up  $\tilde{A} \rightarrow \bar{A}$  such that the quotient  $\tilde{A}/\mathrm{St}(Z)$  exists.*

*Proof.*  $\mathrm{St}(Z)$  is an algebraic subgroup of  $A$ . Hence there is a quotient morphism  $q : A \rightarrow A/\mathrm{St}(Z)$ . Let  $A/\mathrm{St}(Z) \hookrightarrow Z$  be an  $A$ -equivariant smooth compactification. Then  $q$  is a morphism from an Zariski open subset of  $\bar{A}$  to  $Z$  and thus defines a rational

map from  $\bar{A}$  to  $Z$ . Now we just blow up  $\bar{A}$  and  $Z$  to remove the indeterminacies and obtain a regular morphism. Since  $q : A \rightarrow A/\text{St}(Z)$  is equivariant, it is clear that the indeterminacies on  $\bar{A}$  are  $A$ -invariant subvarieties. Therefore the blow-up can be done equivariantly. *Q.E.D.*

### 3.3 Finitely many orbits

We will need the following auxiliary result.

**Lemma 3.12** *Let  $A$  be a semi-abelian variety and  $A \hookrightarrow \bar{A}$  a smooth equivariant algebraic compactification. Then there are only finitely many  $A$ -orbits in  $\bar{A}$ .*

*Proof.* Let  $\tau : \mathbf{C}^n \rightarrow A$  denote the universal covering. Then  $A = \mathbf{C}^n/\Gamma$ , where  $\Gamma = \tau^{-1}\{0\}$ . Note that  $\Gamma$  generates  $\mathbf{C}^n$  as complex vector space.

Let  $H$  be an algebraic subgroup of  $A$ . Then  $H$  is a semi-abelian variety, too. It follows that the connected component  $\hat{H}$  of  $\tau^{-1}(H)$  coincides with the complex vector subspace of  $\mathbf{C}^n$  generated by  $\hat{H} \cap \Gamma$ . Evidently there are only countably many finitely generated subgroups of  $\Gamma$ . It follows that there are only countably many algebraic subgroups  $H$  of  $A$ .

Let  $p$  be a point in  $\bar{A}$  and let  $H = A_p$  be its isotropy group. Let  $Ap$  denote the  $A$ -orbit through  $p$ . Let  $\bar{A}^H$  denote the fixed point set of  $H$ -action, i.e.,  $\bar{A}^H = \{x \in \bar{A} : ax = x, \forall a \in H\}$ . Then  $\bar{A}^H$  is a closed algebraic subvariety of  $\bar{A}$ . Let  $T_p(\bar{A}^H)$  be its Zariski tangent space at  $p$ . Because  $H$  is reductive, the  $H$ -action on  $T_p(\bar{A})$  is almost effective. On the other hand, because  $H$  acts trivially on  $\bar{A}^H$ , the action on  $T_p(\bar{A}^H)$  is likewise trivial. Therefore there is an almost effective  $H$ -action on the quotient vector space  $T_p(\bar{A})/T_p(\bar{A}^H)$ . Since  $H$  is abelian, this implies  $\dim H \leq \dim (T_p(\bar{A})/T_p(\bar{A}^H))$ . From this we deduce

$$\dim(Ap) = \dim A - \dim H \geq \dim X - \dim (T_p(\bar{A})/T_p(\bar{A}^H)) = \dim T_p(\bar{A}^H)$$

Since  $Ap \subset \bar{A}^H$ , it follows that  $\bar{A}^H$  is smooth at  $p$  and  $Ap$  is open in  $\bar{A}^H$ . In particular, there is an open neighborhood  $W$  of  $p$  in  $\bar{A}$  such that  $Ap$  is the only  $A$ -orbit in  $W$  with  $H$  as isotropy group. By virtue of algebraicity it follows that there are only finitely many  $A$ -orbits in  $\bar{A}$  with  $H$  as isotropy group.

Since there are only countably many algebraic subgroups of  $A$ , we obtain as a consequence that there are only countably many  $A$ -orbits in  $\bar{A}$ .

Thus  $A$  is an algebraic group acting on an algebraic variety  $\bar{A}$  with only countably many orbits. This implies that there are actually only finitely many orbits. *Q.E.D.*

### 3.4 Action

Let  $A$  be a semi-abelian variety and let  $\mathbf{P}^N(\mathbf{C})$  be the complex projective  $N$ -space. Then  $A$  acts on the product  $A \times \mathbf{P}^N(\mathbf{C})$  by the group action of the first factor:

$$(a, (b, x)) \in A \times (A \times \mathbf{P}^N(\mathbf{C})) \rightarrow a \cdot (b, x) = (a + b, x) \in A \times \mathbf{P}^N(\mathbf{C}).$$

Let  $p : A \times \mathbf{P}^N(\mathbf{C}) \rightarrow A$  be the first projection. Let  $X$  be an irreducible algebraic subset of  $A \times \mathbf{P}^N(\mathbf{C})$  such that  $p(X) = A$ . We set

$$B = \text{St}(X) = \{a \in A; a \cdot X = X\}^0,$$

and assume that  $\dim B > 0$ . Set  $C = A/B$ .

Taking direct products with  $\mathbf{P}^N(\mathbf{C})$ , one extends the projection  $A \rightarrow C$  to  $\tau : A \times \mathbf{P}^N(\mathbf{C}) \rightarrow C \times \mathbf{P}^N(\mathbf{C})$ . This is a  $B$ -principal bundle. The subvariety  $X$  of  $A \times \mathbf{P}^N(\mathbf{C})$  is  $B$ -invariant; therefore  $X = \tau^{-1}(\tau(X))$ . It follows that  $\tau(X)$  is a closed subvariety of  $C \times \mathbf{P}^N(\mathbf{C})$  which we can regard as the quotient  $X/B$  of  $X$  with respect to the  $B$ -action. In particular  $\pi = \tau|_X : X \rightarrow Y = \tau(X)$  is a  $B$ -principal bundle such that the  $B$ -action on  $X$  is simply the principal right action of  $B$  for this bundle structure.

Let  $\hat{B}$  be a smooth equivariant compactification of  $B$ . Then we have a relative compactification  $\hat{A} \rightarrow C$  of  $A \rightarrow C$  arising as the  $\hat{B}$ -bundle associated to the  $B$ -principal bundle  $A \rightarrow C$ . In other words:  $\hat{A} = A \times_B \hat{B}$  where  $A \times_B \hat{B}$  denotes the quotient of  $A \times \hat{B}$  with respect to the equivalence relation for which  $(a, b) \sim (a', b')$  if and only if there exists an element  $g \in B$  such that  $ag = a'$  and  $b = gb'$ . The projection map  $p$  extends to  $\hat{p} : \hat{A} \times \mathbf{P}^N(\mathbf{C}) \rightarrow \hat{A}$ . Let  $\hat{X}$  be the closure of  $X$  in  $\hat{A}$ . Then  $\hat{X} = X \times_B \hat{B}$ . The compactness of  $\hat{B}$  implies that the projection map  $\hat{\pi} : \hat{X} \rightarrow Y$  is proper.

Let  $E \subset X$  be an irreducible algebraic subset such that

$$(3.13) \quad B \cap \text{St}(E) = \{0\}.$$

**Proposition 3.14** *Let  $\hat{X}$ ,  $X$ ,  $E$ , etc. be as above. Assume in addition that  $E$  is of codimension one, i.e., a divisor. Then there is a  $B$ -equivariant blow-up*

$$\psi : X^\dagger \rightarrow \hat{X}$$

*with center in  $\hat{X} \setminus X$  such that  $X^\dagger$  has a stratification by  $B$ -invariant strata*

$$X^\dagger = \cup_\lambda \Gamma_\lambda$$

*satisfying the following properties:*

- (i)  $\Gamma_\lambda \cong X/B_x$  ( $x \in \Gamma_\lambda$ ) where  $B_x = \{b \in B : b \cdot x = x\}$  is the isotropy group at  $x$ .
- (ii) The closure of  $E$  in  $X^\dagger$  contains none of the strata  $\Gamma_\lambda$ .
- (iii) The open subset  $X$  of  $X^\dagger$  coincides with one of the strata  $\Gamma_\lambda$ .

*Proof.* Before starting the proof we make a remark: Since  $X \rightarrow Y$  is a  $B$ -principal bundle, we can define quotient varieties  $X/H$  for all algebraic subgroups  $H$  of  $B$ . Therefore statement (i) of the proposition makes sense.

Now we start the proof. We will only consider blow-ups  $X^\dagger \rightarrow \hat{X}$  which arise in the following way: We take an equivariant blow-up  $B^\dagger \rightarrow \hat{B}$  and define  $X^\dagger = X \times_B B^\dagger$ . We recall that there are only finitely many  $B$ -orbits in  $B^\dagger$  (Lemma 3.12) and that  $X \times_B B^\dagger$  is defined as a quotient of  $X \times B^\dagger$ . Let  $\{\Omega_\lambda\}_\lambda$  be the family of  $B$ -orbits in  $B^\dagger$ . Then a stratification  $\{\Gamma_\lambda\}_\lambda$  of  $X^\dagger$  is induced as follows: For each  $\lambda$  we define  $\Gamma_\lambda$  is the image of  $X \times \Omega_\lambda$  under the projection  $X \times B^\dagger \rightarrow X \times_B B^\dagger = X^\dagger$ . Each of these  $B$ -orbits  $\Omega_\lambda$  can be written as quotient of  $B$  by some closed algebraic subgroup  $H_\lambda$ :

$$\Omega_\lambda \cong B/H_\lambda.$$

Then  $H_\lambda$  is the isotropy group of the  $B$ -action on  $\Gamma_\lambda$  at any point  $x \in \Gamma_\lambda$  and  $\Gamma_\lambda = X/H_\lambda$ . Thus the stratification  $\{\Gamma_\lambda\}_\lambda$  of  $X^\dagger$  has the properties required by (i), for every choice of an equivariant blow-up  $B^\dagger \rightarrow \hat{B}$ .

By construction, the open subset  $X$  of  $X^\dagger$  coincides with the open  $B$ -orbit in  $B^\dagger$ , hence (iii) follows.

Let us now verify that  $B^\dagger \rightarrow B$  can be chosen in such a way that property (ii) holds, too. For  $y \in Y$  let  $E_y$  be defined as  $E_y = \{p \in E : \pi(p) = y\}$ . We observe that  $\bar{E}_y = \pi^{-1}(y) \cap \bar{E}$  for almost all  $y \in \pi(E)$ . Using [N81], Lemma 4.1., we infer from (3.13) that for a generic point  $y \in \pi(E)$  the fiber  $E_y$  has a discrete stabiliser with respect to the  $B$ -action on  $X$ . Thus we may invoke Proposition 3.10 and deduce that there exists an equivariant blow-up  $B^\dagger \rightarrow \hat{B}$  such that  $E_y$  is generally positioned in  $B^\dagger$ . Let  $X^\dagger \rightarrow \hat{X}$  be the associated blow-up of  $\hat{X}$ . Now  $E_y$  being generally positioned in  $B^\dagger$  implies that the closure of  $E$  in  $X^\dagger$  contains none of the strata  $\Gamma_\lambda$ . *Q.E.D.*

## 4 Second main theorem for jet lifts

Let  $A$  be a semi-abelian variety of dimension  $n$  and let  $T$  be the maximal affine subgroup of  $A$ . Then  $T \cong (\mathbf{C}^*)^t$  and there is an exact sequence of rational homomorphisms

$$0 \rightarrow T \rightarrow A \rightarrow A_0 \rightarrow 0,$$

where  $A_0$  is an abelian variety. Let  $\bar{A}$  be a smooth equivariant compactification of  $A$ . Set  $\partial A = \bar{A} \setminus A$  and let  $J_k(\bar{A}, \log \partial A)$  be the logarithmic  $k$ -jet bundle along  $\partial A$  (cf. [N86]). Then  $A$  acts on  $J_k(\bar{A}, \log \partial A)$  and there is an equivariant trivialization

$$J_k(\bar{A}, \log \partial A) \cong \bar{A} \times J_{k,A},$$

where  $A$  acts trivially on the second factor  $J_{k,A} = \mathbf{C}^{kn}$ . Let  $\bar{J}_{k,A}$  be a projective compactification of  $J_{k,A}$ . With the trivial action of  $A$  on  $\bar{J}_{k,A}$  and the usual action on  $A$  (by translations) and  $\bar{A}$  this yields an  $A$ -equivariant compactification

$$\bar{J}_k(\bar{A}, \log \partial A) = \bar{A} \times \bar{J}_{k,A}$$

of  $J_k(A)$  with an open  $A$ -invariant subset

$$\tilde{J}_k(A) = A \times \bar{J}_{k,A}.$$

For example, we may set  $\bar{J}_{k,A} = \mathbf{P}^{nk}(\mathbf{C})$  or  $\bar{J}_{k,A} = (\mathbf{P}^n(\mathbf{C}))^k$ . Then  $J_k(A) = J_k(\bar{A}, \log \partial A)|_A$  is a Zariski open subset of  $\bar{J}_k(\bar{A}, \log \partial A)$  and

$$J_k(A) \cong A \times J_{k,A}.$$

We set

$$\begin{aligned} J_k^{\text{reg}}(\bar{A}, \log \partial A) &= \{j_k(g) \in J_k(\bar{A}, \log \partial A); j_1(g) \neq 0\} \cong \bar{A} \times J_{k,A}^{\text{reg}}, \\ J_k^{\text{reg}}(A) &= J_k^{\text{reg}}(\bar{A}, \log \partial A)|_A \cong A \times J_{k,A}^{\text{reg}}, \end{aligned}$$

of which elements are called *regular jets*.

Let  $f : \mathbf{C} \rightarrow A$  be a holomorphic curve and  $J_k(f) : \mathbf{C} \rightarrow J_k(A)$  be the  $k$ -jet lift of  $f$ . We denote by  $X_k(f)$  (resp.  $\tilde{X}_k(f)$ ) the Zariski closure of the image  $J_k(f)(\mathbf{C})$  in  $J_k(A)$  (resp.  $\tilde{J}_k(A)$ ):

$$(4.1) \quad X_k(f) \subset J_k(A), \quad \tilde{X}_k(f) \subset \tilde{J}_k(A).$$

**Theorem 4.2** (Second Main Theorem) *Let  $f : \mathbf{C} \rightarrow A$  be an algebraically non-degenerate holomorphic curve. Let  $Z$  be a reduced subvariety of  $X_k(f)$ . Then there exists a natural number  $l_0$  and a compactification  $\bar{X}_k(f)$  of  $X_k(f)$  such that for the closure  $\bar{Z}$  of  $Z$  in  $\bar{X}_k(f)$*

$$(4.3) \quad m_{J_k(f)}(r; \bar{Z}) = S_f(r),$$

$$(4.4) \quad T(r; \omega_{\bar{Z}, J_k(f)}) \leq N_{l_0}(r; J_k(f)^* Z) + S_f(r).$$

In the case of  $k = 0$  the compactification  $\bar{A}$  of  $A$  can be chosen smooth, equivariant, and independent of  $f$ ; moreover, if  $Z$  is a divisor  $D$ , (4.3) and (4.4) take the following forms, respectively:

$$(4.5) \quad m_f(r; \bar{D}) = S_f(r; L(\bar{D})),$$

$$(4.6) \quad T_f(r; L(\bar{D})) \leq N_{l_0}(r; f^*D) + S_f(r; L(\bar{D})).$$

*Proof.* Since the very basic idea of the proof is the same as that of the Main Theorem of [NWY03], it will be helpful to confer it.

We extend the subvariety  $Z$  to the closure in  $\tilde{X}_k(f)$  which is denoted by the same  $Z$ .

We first prove (4.3) and (4.5). Set  $B = \text{St}(X_k(f))$ . Then we have the quotient maps:

$$\begin{aligned} q^B : A &\rightarrow A/B = C, \\ q_k^B : J_k(A) &\rightarrow J_k(A)/B \cong C \times J_{k,A}, \\ \tilde{q}_k^B : \tilde{J}_k(A) &\rightarrow C \times \bar{J}_{k,A}. \end{aligned}$$

By [N98] and [NW03] Lemma 2.3

$$(4.7) \quad \dim B > 0, \quad T_{q_k^B \circ J_k(f)}(r) = S_f(r).$$

Setting  $\tilde{Y}_k = \tilde{X}_k(f)/B$ , we have a quotient map:

$$\tilde{\pi}_k : \tilde{X}_k(f) \rightarrow \tilde{Y}_k \subset C \times \bar{J}_{k,A}.$$

Let  $\bar{B}$  be a smooth equivariant compactification of  $B$ . Define  $\hat{A}$ ,  $\hat{X}_k(f)$ ,  $\hat{Z}$ , etc. as the partial compactifications of  $A$ ,  $\tilde{X}_k(f)$ ,  $Z$ , etc. as in subsection 3.4. We then have proper maps,

$$\begin{aligned} \hat{q}_k^B : \hat{A} \times \bar{J}_{k,A} &\rightarrow C \times \bar{J}_{k,A}, \\ \hat{\pi}_k = \hat{q}_k^B|_{\hat{X}_k(f)} : \hat{X}_k(f) &\rightarrow \tilde{Y}_k \subset C \times \bar{J}_{k,A}, \end{aligned}$$

whose fibers are isomorphic to  $\bar{B}$ .

There are two cases,  $B \subset \text{St}(Z)$  and  $B \not\subset \text{St}(Z)$ , which we consider separately.

(a) Suppose that  $B \subset \text{St}(Z)$ . Set  $\hat{W} = \hat{\pi}_k(\hat{Z}) = \hat{Z}/B$ . Then  $\hat{W}$  has at least codimension one in  $\tilde{Y}_k$ . Let  $T \cong (\mathbf{C}^*)^t$  be the maximal affine subgroup of  $A$  and let  $S$  be that of  $B$ . Then  $S$  is a subgroup of  $T$  and there is a splitting,  $T \cong S \times S'$ . Take an equivariant compactification  $\bar{S}'$  of  $S'$  and set

$$\bar{A} = \hat{A} \times_{S'} \bar{S}'.$$

Then  $\bar{A}$  is an equivariant compactification of  $A$  and  $\hat{A}$ . We have an algebraic exact sequence

$$0 \rightarrow S' \rightarrow C \rightarrow C_0 \rightarrow 0,$$

where  $C_0$  is an abelian variety, and an equivariant compactification  $\bar{C} = C \times_{S'} \bar{S}'$ . Thus  $\hat{q}_k^B$  extends to

$$\bar{q}_k^B : \bar{A} \times J_{k,A} \rightarrow \bar{C} \times J_{k,A},$$

Let  $\bar{X}_k(f)$  (resp.  $\bar{Y}_k, \bar{W}$ ) be the closure of  $\hat{X}_k(f)$  (resp.  $\hat{Y}_k, \hat{W}$ ) in  $\bar{A} \times \bar{J}_{k,A}$  (resp.  $\bar{C} \times \bar{J}_{k,A}$ ). Thus we have the restriction

$$\bar{\pi}_k = \bar{q}_k^B|_{\bar{X}_k(f)} : \bar{X}_k(f) \rightarrow \bar{Y}_k.$$

Note that  $\bar{\pi}_k$  is surjective and

$$(4.8) \quad \bar{W} \neq \bar{Y}_k.$$

It follows from Theorem 2.9 (ii) and (4.7) that

$$(4.9) \quad \begin{aligned} m_{J_k(f)}(r; \bar{Z}) &\leq m_{\bar{\pi}_k \circ J_k(f)}(r; \bar{W}) + O(1) \\ &= O(T_{\bar{\pi}_k \circ J_k(f)}(r)) = S_f(r). \end{aligned}$$

(b) Suppose that  $B \not\subset \text{St}(Z)$ . We set

$$B' = B \cap \text{St}(Z), \quad Z' = Z/B', \quad \tilde{X}'_k(f) = \tilde{X}_k(f)/B', \quad A' = A/B', \quad B'' = B/B'.$$

Moreover, we define  $W$  as the image of  $Z$  under the quotient  $\tilde{X}'_k(f) \rightarrow \tilde{X}'_k(f)/B'' = \tilde{Y}_k$ . We have the following commutative diagram and quotient maps:

$$\begin{array}{ccccc} Z & \xrightarrow[\text{(codim=1)}]{\subsetneq} & \tilde{X}_k(f) & \subset & A \times \bar{J}_{k,A} \\ \downarrow & & \downarrow & & \downarrow q_k^{B'} \\ Z' & \xrightarrow[\text{(codim=1)}]{\subsetneq} & \tilde{X}'_k(f) & \subset & A' \times \bar{J}_{k,A} \\ \downarrow \tilde{\pi}'_k|_{Z'} & & \downarrow \tilde{\pi}'_k & & \downarrow q_k^{B''} \\ W & \subset & \tilde{Y}_k & \subset & C \times \bar{J}_{k,A} \end{array}$$

Note that

$$(4.10) \quad \text{St}(X'_k(f)) = B'', \quad \text{St}(Z') \cap B'' = \{0\}.$$

Let  $\bar{B}''$  be a smooth equivariant compactification of  $B''$ . We have

$$\begin{aligned}\hat{A}' &= A' \times_{B''} \bar{B}'', \\ \hat{\partial}A' &= \hat{A}' \setminus A', \\ \hat{X}'_k(f) &= \tilde{X}'_k(f) \times_{B''} \bar{B}'', \\ \hat{Z}' &= \bar{Z}' \quad (\text{the closure of } Z' \text{ in } \hat{X}'_k(f)), \\ \hat{\partial}X'_k(f) &= \hat{X}'_k(f) \setminus \tilde{X}'_k(f).\end{aligned}$$

Note that the boundary divisor  $\hat{\partial}A'$  has only normal crossings (Proposition 3.9 (i)). We obtain proper maps

$$\begin{array}{ccccc}\hat{Z}' & \subsetneq & \hat{X}'_k(f) & \subset & \hat{A}' \times \bar{J}_{k,A} \\ \downarrow \hat{\pi}'_k|_{\hat{Z}'} & & \downarrow \hat{\pi}'_k & & \downarrow \hat{q}_k^{B''} \\ \hat{W} & \subset & \tilde{Y}_k & \subset & C \times \bar{J}_{k,A},\end{array}$$

where  $\hat{W} = \hat{\pi}'_k(\hat{Z}')$ . By Proposition 3.14 we have a blow-up

$$\psi : \hat{X}_k'^{\dagger}(f) \rightarrow \hat{X}'_k(f)$$

with center in  $\hat{\partial}X'_k(f)$ , the strict transform  $\hat{Z}'^{\dagger}$  of  $\hat{Z}'$  and the boundary

$$\Gamma = \hat{X}_k'^{\dagger}(f) \setminus \tilde{X}'_k(f)$$

with stratification  $\Gamma = \cup_{\lambda} \Gamma_{\lambda}$  such that

$$(4.11) \quad \Gamma_{\lambda} \cong \tilde{X}'_k(f)/\text{Iso}_x(B'') \quad (x \in \Gamma_{\lambda}),$$

$$(4.12) \quad \Gamma_{\lambda} \cap \hat{Z}'^{\dagger} \neq \Gamma_{\lambda}.$$

Here, if  $k = 0$ , we use Proposition 3.10 in place of Proposition 3.14, and deduce the stated property for  $\bar{A}$ .

Let  $\psi_{*l} : J_l(\hat{X}_k'^{\dagger}(f), \log \Gamma) \rightarrow J_l(\hat{X}'_k(f), \log \hat{\partial}X'_k(f))$  be the morphism naturally induced by  $\psi$ . We consider a sequence of morphisms

$$\begin{aligned}J_l(\hat{Z}'^{\dagger}, \log \Gamma) &\subset J_l(\hat{X}_k'^{\dagger}(f), \log \Gamma) \xrightarrow{\psi_{*l}} J_l(\hat{X}'_k(f), \log \hat{\partial}X'_k(f)) \\ &\hookrightarrow J_l(\hat{A}' \times \bar{J}_{k,A}, \log(\hat{\partial}A' \times \bar{J}_{k,A})) \\ &\cong J_l(\hat{A}', \log \hat{\partial}A') \times J_l(\bar{J}_{k,A}) \\ &\cong \hat{A}' \times J_l(J_{k,A'}) \times J_l(\bar{J}_{k,A}) \\ &\xrightarrow{\text{proj.}} J_l(J_{k,A'}) \times J_l(\bar{J}_{k,A}).\end{aligned}$$

Thus we have a morphism

$$\beta_l : J_l(\hat{X}_k'^{\dagger}(f), \log \Gamma) \rightarrow J_l(J_{k,A'}) \times J_l(\bar{J}_{k,A}).$$

Let  $p_l : J_l(\hat{X}_k'^{\dagger}(f)) \rightarrow \hat{X}_k'^{\dagger}(f)$  be the projection to the base space. Henceforth we obtain a proper morphism

$$\gamma_l = (\hat{\pi}_k' \circ \psi \circ p_l) \times \beta_l : J_l(\hat{X}_k'^{\dagger}(f), \log \Gamma) \rightarrow \tilde{Y}_k \times J_l(J_{k,A'}) \times J_l(\bar{J}_{k,A}).$$

We claim that for some  $l_0 \geq 1$

$$\text{Claim 4.13} \quad \gamma_{l_0}(J_{l_0}(\hat{Z}')) \neq \gamma_{l_0}(J_{l_0}(\hat{X}_k'(f))).$$

Assume contrarily that  $\gamma_l(J_l(\hat{Z}')) = \gamma_l(J_l(\hat{X}_k'(f)))$  for all  $l \geq 1$ . Then for an arbitrary  $z \in \mathbf{C}$

$$(4.14) \quad J_l(q_1^{B'} \circ J_k(f))(z) \in \gamma_l(J_l(\hat{Z}'^{\dagger}, \log \Gamma)).$$

Fix  $z_0 \in \mathbf{C}$ . Then  $\hat{\pi}_k \circ J_k(f)(z_0) \in \tilde{Y}_k$  and we set

$$\xi_l = J_l(q_1^{B'} \circ J_k(f))(z_0) \in \gamma_l(J_l(\hat{Z}'^{\dagger}, \log \Gamma)), \quad l \geq 1.$$

Set  $\Xi_l = \gamma_l^{-1}(\xi_l)$  for  $l \geq 0$ . Then the restriction  $p_l|_{\Xi_l}$  is proper and  $p_l|_{\Xi_l} : \Xi_l \rightarrow p_l(\Xi_l)$  is an isomorphism. We set

$$\Lambda_l = p_l(\Xi_l), \quad l = 1, 2, \dots$$

The sequence of  $\Lambda_l \supset \Lambda_{l+1}$ ,  $l = 1, 2, \dots$  terminates to  $\Lambda_{\infty} = \Lambda_{l_0} = \Lambda_{l_0+1} = \dots (\subset \hat{X}_k'^{\dagger}(f))$  for some  $l_0$ . Then  $\Lambda_{\infty} \neq \emptyset$ . If  $\Lambda_{\infty} \cap \tilde{X}_k'(f) \neq \emptyset$ , there is an element  $a \in A'$  such that

$$a \cdot (J_l(q_1^{B'} \circ J_k(f))(z_0)) \in J_l(Z'), \quad \forall l \geq 0.$$

By the identity principle we deduce that  $a \cdot \tilde{X}_k'(f) \subset Z'$ ; this is absurd.

Now assume that  $\Lambda_{\infty} \cap \Gamma \neq \emptyset$ . There is a point  $x_0 \in \Lambda_{\infty} \cap \Gamma$  such that

$$(x_0, \xi_l) \in J_l(\hat{Z}'^{\dagger})_{x_0}, \quad l \geq 1.$$

Let  $\Gamma_{\lambda_0}$  be the boundary stratum containing  $x_0$ . Let  $\alpha : \tilde{X}_k'(f) \rightarrow \tilde{X}_k'(f)/\text{Iso}_{x_0}(B'') \cong \Gamma_{\lambda_0}$  be the quotient map. Then there exists an element  $a_0 \in A$  such that

$$a \cdot (\alpha \circ q_1^{B'} \circ J_k(f)(z)) \in \Gamma_{\lambda_0} \cap \hat{Z}'^{\dagger}$$

in a neighborhood of  $z_0$  and hence for all  $z \in \mathbf{C}$ . Henceforth a contradiction follows from this, (4.12) and the image  $J_k(f)(\mathbf{C})$  being Zariski dense in  $X_k(f)$ .

This proves Claim 4.13.

We infer (4.4) and (4.6) as in the proof of the Main Theorem of [NWY02] p. 152 (cf. [NWY02] (5.12)) with a modification as follows. Let  $\bar{X}_k(f) = \bigcup_{\alpha} U_{\alpha}$  be a finite affine covering, and let  $\sigma_{\alpha\nu}$  be the defining functions of  $Z \cap U_{\alpha}$ . Then by Claim 4.13 there is a rational function  $\eta$  on  $\tilde{Y}_k \times J_{l_0}(J_{k,A'} \times J_{l_0}(\bar{J}_{k,A}))$ , regarded as a rational function on  $J_{l_0}(X_k(f))$  such that

$$\begin{aligned} \eta \circ J_{l_0}(q_1^{B'} \circ J_k(f))(z) &\neq 0, \quad z \in \mathbf{C}, \\ \eta|_{U_{\alpha}} &= \sum_{\nu} \sum_{0 \leq j \leq l_0} a_{\alpha\nu j} d^j \sigma_{\alpha\nu}, \end{aligned}$$

where the coefficients  $a_{\alpha\nu j}$  are jet differentials on  $U_{\alpha}$ . Then we have

$$\begin{aligned} \eta|_{U_{\alpha}} &= \sum_{\nu} \sigma_{\alpha\nu} \sum_{0 \leq j \leq l_0} a_{\alpha\nu j} \frac{d^j \sigma_{\alpha\nu}}{\sigma_{\alpha\nu}}, \\ |\eta|_{U_{\alpha}} &\leq \left( \sum_{\nu} |\sigma_{\alpha\nu}|^2 \right)^{1/2} \left( \sum_{\nu} \left( \sum_{0 \leq j \leq l_0} |a_{\alpha\nu j}| \left| \frac{d^j \sigma_{\alpha\nu}}{\sigma_{\alpha\nu}} \right| \right)^2 \right)^{1/2}, \\ \frac{1}{(\sum_{\nu} |\sigma_{\alpha\nu}|^2)^{1/2}} &\leq \frac{1}{|\eta|_{U_{\alpha}}} \left( \sum_{\nu} \left( \sum_{0 \leq j \leq l_0} |a_{\alpha\nu j}| \left| \frac{d^j \sigma_{\alpha\nu}}{\sigma_{\alpha\nu}} \right| \right)^2 \right)^{1/2}. \end{aligned}$$

Therefore we deduce from (4.7) that

$$\begin{aligned} m_{J_k(f)}(r; \bar{Z}) &= S_f(r), \\ N(r; J_k(f)^* Z) &= N_{l_0}(r; J_k(f)^* Z) + S_f(r). \end{aligned}$$

Combining these with the First Main Theorem 2.9, we obtain (4.3) and (4.4).

Let us now prove the additional statements for the case  $k = 0$ . In this case we take the quotient,  $q : A \rightarrow A/\text{St}(Z)$  and we deal with the holomorphic curve  $q \circ f : \mathbf{C} \rightarrow A/\text{St}(Z)$  and the subvariety  $Z/\text{St}(Z)$ . In this way it is reduced to the case when  $\text{St}(Z) = \{0\}$ . Then the compactification of  $A$  due to Proposition 3.10 works for an arbitrary algebraically nondegenerate  $f : \mathbf{C} \rightarrow A$ .

If  $Z$  is a divisor  $D$  on  $A$ , then Proposition 3.9 (ii) implies that  $D$  is big and we can deduce (4.5) with the help of Lemma 2.4. *Q.E.D.*

## 5 Higher codimensional subvarieties of $X_k(f)$

Let  $f : \mathbf{C} \rightarrow A$  be a holomorphic curve in a semi-abelian variety  $A$ . We use the same notation,  $X_k(f)$ ,  $\text{St}(X_k(f))$ , etc. as in the previous section.

The purpose of this section is to prove the following.

**Theorem 5.1** *Let  $f : \mathbf{C} \rightarrow A$  be a holomorphic curve and let  $Z \subset X_k(f)$  be a subvariety of  $\text{codim}_{X_k(f)} Z \geq 2$ . Then there is a compactification  $\bar{X}_k(f)$  such that for the closure  $\bar{Z}$  of  $Z$  in  $\bar{X}_k(f)$*

$$T(r; \omega_{\bar{Z}J_k(f)}) \leq \epsilon T_f(r) ||_{\epsilon}, \quad \forall \epsilon > 0.$$

In particular,

$$(5.2) \quad N(r; J_k(f)^* Z) \leq \epsilon T_f(r) ||_{\epsilon}, \quad \forall \epsilon > 0.$$

*Remark.* (i) For an abelian variety  $A$  this was proved by [Y04].

(ii) As a consequence, estimate (5.2) is independent of the choice of the compactification  $\bar{X}_k(f)$ .

It suffices to prove Theorem 5.1 for irreducible  $Z$ . Hence, we assume throughout this section that  $Z$  is *irreducible*.

Our proof naturally divides into three steps (a)~(c). Before going to discuss the details, we give an outline of the proof.

(a) First, we reduce the case to the one that  $A$  admits a splitting  $A = B \times C$  where  $B$  and  $C$  are semi-abelian varieties such that

$$(5.3) \quad B \subset \text{St}(X_l(f)) \quad \text{for all } l \geq 0$$

and the composition of  $f$  and the second projection  $q^B : A \rightarrow A/B = C$  satisfies

$$(5.4) \quad T_{q^B \circ f}(r) = S_f(r).$$

By this reduction, we may assume that the variety  $X_l(f)$  has splitting  $X_l(f) = B \times (X_l(f)/B)$  for all  $l \geq 0$ .

We also make a reduction such that the image of  $Z$  under the second projection  $\pi_k : X_k(f) \rightarrow X_k(f)/B$  has a Zariski dense image. Hence by the assumption  $\text{codim}_{X_k(f)} Z \geq 2$ , we may assume  $\text{codim}_{\pi_k^{-1}(x)} Z \cap \pi_k^{-1}(x) \geq 2$  for general  $x \in X_k(f)/B$ .

(b) The second step is the main part of the proof. Using the above reduction, we shall construct auxiliary divisors  $F_l \subset \bar{B} \times (X_{k+l}(f)/B)$  for all  $l \geq 0$  with the following properties:

$$(i) \quad (l+1)N_1(r; J_k(f)^* Z) \leq N(r; J_{k+l}(f)^* F_l) + \epsilon T_f(r) ||_{\epsilon}, \quad \forall \epsilon > 0:$$

$$(ii) \quad T_{J_{k+l}(f)}(r; L(F_l)) \leq n(l)T_{\gamma \circ f}(r; D_B) + \epsilon T_f(r; D) ||_{\epsilon}, \quad \forall \epsilon > 0,$$

where  $\gamma : A \rightarrow B$  is the first projection,  $D$  is an ample line bundle over  $\bar{A}$ ,  $D_B$  is an ample line bundle over  $\bar{B}$  and  $n(l)$  is a positive integer such that  $\lim_{l \rightarrow \infty} n(l)/l = 0$ .

(c) Finally, by (i) and (ii) above we have

$$\begin{aligned} N_1(r; J_k(f)^*Z) &\leq \frac{1}{l+1}N(r; J_{k+l}(f)^*F_l) + \frac{\epsilon}{l+1}T_f(r; D)||_\epsilon \\ &\leq \frac{n(l)}{l+1}T_{\gamma \circ f}(r; D_B) + \frac{\epsilon}{l+1}T_f(r; D)||_\epsilon \end{aligned}$$

for all  $\epsilon > 0$  and all integer  $l \geq 0$ . Since  $n(l)/l \rightarrow 0$  ( $l \rightarrow \infty$ ), we have

$$N_1(r; J_k(f)^*Z) \leq \epsilon(T_{\gamma \circ f}(r; D_B) + T_f(r; D)||_\epsilon, \quad \forall \epsilon > 0.$$

Since  $T_{\gamma \circ f}(r; D_B) = O(T_f(r; D))$ , the proof is completed.

**(a) Reduction.** Let  $f : \mathbf{C} \rightarrow A$  be as above. Let  $I_k : \hat{X}_k(f) (\hookrightarrow A \times J_{k,A}) \rightarrow J_{k,A}$  be the jet projection. It follows from [N77] (or [NWY02] Lemma 3.8) that

$$(5.5) \quad T_{I_k \circ J_k(f)}(r) = S_f(r).$$

We need the following.

**Lemma 5.6** *Let the notation be as above. Let  $G = \cap_{l \geq 0} \text{St}(X_l(f))$  and let  $q^G : A \rightarrow A/G$  be the quotient map. Then*

$$T_{q^G \circ f}(r) = O(T_{I_k \circ J_k(f)}(r)) (= S_f(r)).$$

*Proof.* This is essentially the same as (4.7) and follows from the jet projection method; cf. [NW03] Lemma 2.4, [NWY02] Lemma 3.8 and their proofs. *Q.E.D.*

**Lemma 5.7** *Let  $B \subset A$  be a semi-abelian subvariety. Put  $B' = B \cap (\cap_{l \geq 0} \text{St}(X_l(f)))$ . Let  $q^B : A \rightarrow A/B$  and  $q^{B'} : A \rightarrow A/B'$  be quotient mappings. Then we have*

$$T_{q^{B'} \circ f}(r) = O(T_{q^B \circ f}(r)) + S_f(r).$$

*Proof.* We write  $G = \cap_{l \geq 0} \text{St}(X_l(f))$ . Taking the natural embedding  $A/B' \rightarrow (A/B) \times (A/G)$ , we see that

$$T_{q^{B'} \circ f}(r) = O(T_{q^B \circ f}(r) + T_{q^G \circ f}(r)).$$

Thus the claim follows from Lemma 5.6. *Q.E.D.*

**Lemma 5.8** *Let  $A$  and  $A'$  be semi-abelian varieties with a surjective homomorphism  $p : A \rightarrow A'$ . Let  $g : \mathbf{C} \rightarrow A'$  be a holomorphic curve. Then we have a holomorphic curve  $\hat{g} : \mathbf{C} \rightarrow A$  such that  $p \circ \hat{g} = g$  and*

$$T_{\hat{g}}(r) = O(T_g(r)).$$

*Proof.* Set  $n = \dim A$  and  $n' = \dim A'$ . Let  $\varpi : \tilde{A} \cong \mathbf{C}^n \rightarrow A$  and  $\tilde{A}' \cong \mathbf{C}^{n'} \rightarrow A'$  be the universal covering. Then there is a surjective linear homomorphism  $\tilde{p} : \tilde{A} \rightarrow \tilde{A}'$ . Let  $\tilde{g} : \mathbf{C} \rightarrow \tilde{A}'$  be the lifting of  $g$ . Let  $g(z) = \sum_{j=1}^{n'} g_j(z) e'_j$  with basis  $\{e'_j\}$  of  $\tilde{A}'$ . Take a basis  $\{e_j\}$  of  $\tilde{A}$  such that  $\tilde{p}(e_j) = e'_j$ ,  $1 \leq j \leq n'$ . Then we set  $\hat{g}(z) = \varpi(\sum_{j=1}^{n'} g_j(z) e_j)$ . It immediately follows from the definition of order functions (see [NWY02] §3) that  $\hat{g}$  satisfies the requirement. *Q.E.D.*

Now we are going to reduce our proof to the case such that  $A = B \times C$  and that  $B$  and  $C$  are semi-abelian subvarieties satisfying (5.3) and (5.4). Let  $\mathcal{B}$  be the set of all semi-abelian subvarieties  $B \subset A$  such that

$$T_{q^B \circ f}(r) = S_f(r).$$

Then since  $\cap_{l \geq 0} \text{St}(X_l(f)) \in \mathcal{B}$ , we have  $\mathcal{B} \neq \emptyset$ . Let  $B \in \mathcal{B}$  be a minimal element of  $\mathcal{B}$ ; i.e., if  $B' \subset B$  and  $B' \in \mathcal{B}$ , then  $B' = B$ . If  $B_i \in \mathcal{B}$ ,  $i = 1, 2$ , we deduce from Lemma 5.7 that  $B_1 \cap B_2 \in \mathcal{B}$ . Thus we get

$$B \subset \cap_{l \geq 0} \text{St}(X_l(f)).$$

Put  $C = A/B$  and let  $q^B : A \rightarrow C$  be the quotient map. By Lemma 5.8 we may take a holomorphic curve  $g : \mathbf{C} \rightarrow A$  such that  $q^B \circ g = q^B \circ f$  and

$$(5.9) \quad T_g(r) = S_f(r).$$

We may assume that the Zariski closure of the image  $g(\mathbf{C})$  is a semi-abelian subvariety  $C' \subset A$  ([N77], [N81]). Define the semi-abelian variety  $\tilde{A}$  by the following pull-back.

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{p_2} & A \\ p_1 \downarrow & & \downarrow q^B \\ C' & \xrightarrow{q^B|_{C'}} & C \end{array}$$

Then  $\tilde{A} = \{(c, a) \in C' \times A : q^B(c) = q^B(a)\}$ . The inclusion map  $i : C' \rightarrow A$  yields a map  $\tau : C' \rightarrow \tilde{A}$  defined by  $\tau(x) = (x, i(x))$ . Note that this morphism  $\tau$  is a section for  $p_1 : \tilde{A} \rightarrow C'$ . Hence this bundle is trivial, i.e.  $\tilde{A} \cong B \times C'$  and  $\tilde{A}/B = C'$ .

Put  $\tilde{f} = g \times f : \mathbf{C} \rightarrow \tilde{A}$ . Then by (5.9) we have

$$(5.10) \quad T_f(r) = O(T_{\tilde{f}}(r)), \quad T_{\tilde{f}}(r) = O(T_f(r)),$$

$$(5.11) \quad T_{p_1 \circ \tilde{f}}(r) = S_{\tilde{f}}(r).$$

Put

$$(5.12) \quad B' = B \cap \left( \cap_{l \geq 0} \text{St}(X_l(\tilde{f})) \right)$$

and  $p'_1 : \tilde{A} \rightarrow \tilde{A}/B'$  be the quotient map. By Lemma 5.7 and (5.11), we have

$$(5.13) \quad T_{p'_1 \circ \tilde{f}}(r) = S_{\tilde{f}}(r).$$

Put  $q^{B'} : A \rightarrow A/B'$  be the quotient map. Then we have

$$(5.14) \quad T_{q^{B'} \circ f}(r) = O(T_{p'_1 \circ \tilde{f}}(r)).$$

Hence by (5.10), (5.13) and (5.14) we conclude  $B' \in \mathcal{B}$ . Since  $B$  is minimal in  $\mathcal{B}$ , we get  $B' = B$ . By (5.12) we have  $B \subset \cap_{l \geq 0} \text{St}(X_l(\tilde{f}))$ . Let  $p_{2,k} : X_k(\tilde{f}) \rightarrow X_k(f)$  be the morphism induced from  $p_2 : \tilde{A} \rightarrow A$ . Set

$$\tilde{Z} = p_{2,k}^{-1}(Z) \subset X_k(\tilde{f}).$$

Note that

$$N_1(r; J_k(f)^*Z) = N_1(r; J_k(\tilde{f})^*\tilde{Z})$$

and that (5.10) holds.

For the reduction we need  $\text{codim}_{X_k(\tilde{f})} \tilde{Z} \geq 2$ . By Lemma 5.7 we see that

$$B \subset (\cap_{l \geq 0} \text{St}(X_l(f))) \cap (\cap_{l \geq 0} \text{St}(X_l(\tilde{f}))).$$

Thus  $p_{2,l} : X_l(\tilde{f}) \rightarrow X_l(f)$  is  $B$ -equivariant, and induces a morphism

$$p_{2,l}^B : X_l(\tilde{f})/B \rightarrow X_l(f)/B.$$

Let  $\pi_l : X_l(f) \rightarrow X_l(f)/B$  be the quotient map. Then it follows from (5.4) and (5.5) that

$$(5.15) \quad T_{\pi_l \circ J_l(f)}(r) = S_f(r).$$

If the image  $\pi_k(Z)$  is not Zariski dense in  $X_k(f)/B$ , there is a Cartier divisor  $H$  on  $X_k(f)/B$  containing  $\pi_k(Z)$ . Then, making use of (5.15) and the natural embedding  $X_k(f)/B \hookrightarrow (A/B) \times J_{k,A}$  we get

$$(5.16) \quad \begin{aligned} N_1(r; J_k(f)^*Z) &\leq N(r; (\pi_k \circ J_k(f))^*H) = O(T_{\pi_k \circ J_k(f)}(r)) \\ &= S_f(r). \end{aligned}$$

Therefore the proof of Theorem 5.1 is finished in this case.

We assume that  $\pi_k(Z)$  is Zariski dense in  $X_k(f)$ , and has a relative dimension at most  $\dim B - 2$ . Therefore the relative dimension of  $\tilde{Z} \rightarrow X_k(\tilde{f})/B$  is at most  $\dim B - 2$ , and hence  $\text{codim}_{X_k(\tilde{f})} \tilde{Z} \geq 2$ .

Hence, by replacing  $A$  by  $\tilde{A}$ ,  $C$  by  $C'$ ,  $f$  by  $\tilde{f}$  and  $Z$  by  $p_2^{-1}(Z)$ , we may reduce our problem to the desired situation (5.3) and (5.4).

Therefore we assume the following in the sequel:

(i) Let  $B \subset A$  be a semi-abelian subvariety satisfying

$$(5.17) \quad B \subset \cap_{l \geq 0} \text{St}(X_l(f)),$$

$$(5.18) \quad T_{q^B \circ f}(r) = S_f(r),$$

$$(5.19) \quad A \cong B \times (A/B),$$

where  $q^B : A \rightarrow A/B$  is the quotient map.

(ii)  $\pi_k(Z)$  is Zariski dense in  $X_k(f)/B$ .

**(b) Auxiliary divisor.** Let the notation and the assumption be as above. Set  $C = A/B$ . We have

$$(5.20) \quad A \cong B \times C.$$

Then it naturally induces

$$X_l(f) \cong B \times (X_l(f)/B) \quad (l \geq 0).$$

Let  $\bar{B}$  be an equivariant compactification of  $B$  and set  $\hat{X}_l(f) = \bar{B} \times (X_l(f)/B)$ . Let

$$\begin{aligned} \hat{\gamma}_l : \hat{X}_l(f) &\rightarrow \bar{B}, \\ \hat{\pi}_l : \hat{X}_l(f) &\rightarrow X_l(f)/B \end{aligned}$$

be the natural projections.

We denote by  $Z^{\text{ns}}$  the set of non-singular points of  $Z$ .

**Lemma 5.21** *Let  $L \rightarrow \bar{B}$  be an ample line bundle. Then there is a sequence of natural numbers  $n(1), n(2), n(3), \dots$  satisfying the following:*

$$(i) \quad \lim_{l \rightarrow \infty} \frac{n(l)}{l} = 0.$$

(ii) *There exist effective Cartier divisors  $F_l \subset \hat{X}_{k+l}(f)$  and line bundles  $M_l$  on  $X_{k+l}(f)/B$  such that  $F_l$  is defined by a non-zero element of*

$$H^0(\hat{X}_{k+l}(f), \hat{\gamma}_{k+l}^* L^{\otimes n(l)} \otimes (\hat{\pi}_{k+l})^* M_l)$$

*and that for every point  $a \in \mathbf{C}$  with  $J_k(f)(a) \in Z^{\text{ns}}$*

$$\text{ord}_a J_{k+l}(f)^* F_l \geq l + 1.$$

*Proof.* Let  $f_B : \mathbf{C} \rightarrow B$  be the holomorphic curve defined by the composition of  $f$  and the first projection  $A \rightarrow B$ . Let  $f_C : \mathbf{C} \rightarrow C$  be the holomorphic curve defined by the composition of  $f$  and the second projection  $A \rightarrow C$ . Then  $f_B$  and  $f_C$  have Zariski-dense images. Let  $l \geq 0$  be an integer, let  $p_{k+l,k} : J_{k+l,A} \rightarrow J_{k,A}$  be the natural projection, and let

$$T \subset J_{k+l}(A) \times C \times J_{k,A} \cong B \times C \times J_{k+l,A} \times C \times J_{k,A}$$

be the Zariski closed subset defined by

$$T = \{(b, c, v, c', v') \in B \times C \times J_{k+l,A} \times C \times J_{k,A}; b = 0, c = c', v' = p_{k+l,k}(v)\}.$$

Let  $\lambda : B \times C \times J_{k+l,A} \times C \times J_{k,A} \rightarrow C \times J_{k+l,A}$  be the product of the second projection and the third projection. We recall the following from [Y04] Proposition 2.1.1.

**Lemma 5.22** *There exists a closed subscheme  $\mathcal{T} \subset J_{k+l}(A) \times C \times J_{k,A}$  with the following properties:*

- (i)  $\text{Supp } \mathcal{T} = T$ .
- (ii) *The restriction  $\lambda' = \lambda|_{\mathcal{T}} : \mathcal{T} \rightarrow C \times J_{k+l,A}$  is a finite morphism. Furthermore the restriction of the direct image sheaf  $\lambda'_*(\mathcal{O}_{\mathcal{T}})$  to  $C \times J_{k+l,A}^{\text{reg}}$  is a rank  $l+1$  locally free  $\mathcal{O}_{C \times J_{k+l,A}^{\text{reg}}}$ -module.*
- (iii) *Let  $f : \mathbf{C} \rightarrow A$  be a holomorphic curve such that  $f_B(a) = 0$ . Then*

$$\text{ord}_a J_{k+l}(f)^* \mathcal{T}_{\rho \circ J_k(f)(a)} \geq l+1.$$

Let  $r_1 : Z^\dagger \rightarrow \bar{Z}$  be a desingularization of  $\bar{Z}$  such that  $r_1$  gives an isomorphism over  $Z^{\text{ns}}$ . Put  $Y_k = X_k(f)/B$ . Consider the sequence of morphisms

$$(5.23) \quad Z^{\text{ns}} \xrightarrow{r_0} Z^\dagger \xrightarrow{r_1} \bar{Z} \xrightarrow{r_2} \hat{X}_k(f) \xrightarrow{\hat{\pi}_k} Y_k.$$

Here  $r_0, r_1 \circ r_0$  are open immersions and  $r_2$  is a closed immersion. Put the composition of morphisms to be  $r = \hat{\pi}_k \circ r_2 \circ r_1 : Z^\dagger \rightarrow Y_k$ . Let  $Y_k^{\text{fl}}$  be a Zariski open subset of  $Y_k$  such that  $Y_k^{\text{fl}}$  is non-singular and the fibers of  $r : Z^\dagger \rightarrow Y_k$  over  $Y_k^{\text{fl}}$  are all of the same dimension  $\dim Z^\dagger - \dim Y_k$ . Then the restriction of the family  $r : Z^\dagger \rightarrow Y_k$  to  $Y_k^{\text{fl}}$  is a flat family.

Consider the pull back of the sequence of morphisms (5.23) by the natural projection  $B \times Y_k \rightarrow Y_k$ :

$$B \times Z^{\text{ns}} \xrightarrow{s_0} B \times Z^\dagger \xrightarrow{s_1} B \times \bar{Z} \xrightarrow{s_2} B \times \hat{X}_k(f) \xrightarrow{s_3} B \times Y_k.$$

Again put the composition of these morphisms to be  $s = s_3 \circ s_2 \circ s_1 : B \times Z^\dagger \rightarrow B \times Y_k$ . Then  $s$  maps as

$$s : (a, z) \in B \times Z^\dagger \rightarrow (a, r(z)) \in B \times Y_k.$$

Let  $L$  be an ample line bundle on  $\bar{B}$  and set

$$(5.24) \quad \phi : (a, w) \in B \times \hat{X}_k(f) \rightarrow a + \gamma_k(w) \in \bar{B}.$$

Let  $L_1^\dagger$  be the line bundle on  $B \times Z^\dagger$  which is the pull back of  $L$  by the composition of morphisms

$$B \times Z^\dagger \xrightarrow{s_2 \circ s_1} B \times \hat{X}_k(f) \xrightarrow{\phi} \bar{B}.$$

Since the restriction of  $s$  over  $B \times Y_k^{\text{fl}}$  (i.e.,  $s|_{B \times Y_k^{\text{fl}}} : B \times (Z^\dagger|_{Y_k^{\text{fl}}}) \rightarrow B \times Y_k^{\text{fl}}$ ) is a flat family, the semi-continuity theorem [H77] p. 288 implies that there is a Zariski open subset  $U_n \subset B \times Y_k^{\text{fl}}$  ( $n > 0$ ) such that  $H^0((B \times Z^\dagger)|_P, L_{1,P}^{\dagger \otimes n})$  are all the same dimensional  $\mathbf{C}$ -vector spaces for  $P \in U_n$ . Put this dimension as  $G_n$ . Here  $(B \times Z^\dagger)|_P$  denotes the fiber of the morphism  $s : B \times Z^\dagger \rightarrow B \times Y_k$  over  $P \in B \times Y_k$ , and  $L_{1,P}^{\dagger \otimes n}$  is the induced line bundle. Since the intersection  $\cap_{n \geq 1} U_n$  is non-empty, put  $(a, w) \in \cap_{n \geq 1} U_n$  and replacing  $L$  by the pull back by the morphism

$$B \ni x \mapsto x + a \in B$$

we may assume  $a = 0 \in B$ .

Now for a positive integer  $l > 0$ , let  $\mathcal{T}_l^\dagger \subset A \times J_{k+l,A} \times C \times J_{k,A}$  be the closed subscheme, and let  $\lambda : \mathcal{T}_l^\dagger \rightarrow C \times J_{k+l,A}$  be the morphism obtained in Lemma 5.22. Then  $\lambda$  has the following properties;

- (i)  $\lambda$  is finite,
- (ii) the direct image sheaf  $\lambda_* \mathcal{O}_{\mathcal{T}_l^\dagger}$  is locally generated by  $l + 1$  elements as  $\mathcal{O}_{C \times J_{k+l,A}}$  module on  $C \times J_{k+l,A}^{\text{reg}}$ ,
- (iii)  $\lambda$  induces an isomorphism of the underlying topological spaces of  $\mathcal{T}_l^\dagger$  and  $C \times J_{k+l,A}$ .

Since  $Y_{k+l}$  is a Zariski closed subset of  $C \times J_{k+l,A}$ , we denote  $\sigma_{k+l} : Y_{k+l} \rightarrow C$  for the composition with the first projection  $C \times J_{k+l,A} \rightarrow C$  and denote  $\eta_{k+l} : Y_{k+l} \rightarrow J_{k+l,A}$  for the composition with the second projection. We have the closed immersion

$$(5.25) \quad B \times Y_{k+l} \times Y_k \subset B \times C \times J_{k+l,A} \times C \times J_{k,A} \cong A \times J_{k+l,A} \times C \times J_{k,A},$$

where the first inclusion is given by

$$B \times Y_{k+l} \times Y_k \ni (b, v, v') \mapsto (b, \sigma_{k+l}(v), \eta_{k+l}(v), \sigma_k(v'), \eta_k(v')) \in B \times C \times J_{k+l,A} \times C \times J_{k,A}$$

and the second identification is given by

$$B \times C \times J_{k+l,A} \times C \times J_{k,A} \ni (b, c, u, c', u') \mapsto ((b, c), u, c', u') \in A \times J_{k+l,A} \times C \times J_{k,A}.$$

Let  $\mathcal{S}_l \subset B \times Y_{k+l} \times Y_k$  be the closed subscheme obtained by the pull-back of  $\mathcal{T}_l^\dagger$  by (5.25).

Let  $q : \mathcal{S}_l \rightarrow Y_{k+l}$  be the composition with the second projection  $B \times Y_{k+l} \times Y_k \rightarrow Y_{k+l}$ .

We put

$$Y_{k+l}^{\text{reg}} = Y_{k+l} \cap (C \times J_{k+l,A}^{\text{reg}}),$$

which is the Zariski open subset of  $Y_{k+l}$ . Then by the above properties of  $\lambda$ , we have the corresponding properties for  $q$ ;

- (i)  $q$  is finite,
- (ii) the direct image sheaf  $q_* \mathcal{O}_{\mathcal{S}_l}$  is locally generated by  $l+1$  elements as  $\mathcal{O}_{Y_{k+l}}$ -module on  $Y_{k+l}^{\text{reg}}$ ,
- (iii)  $q$  gives the isomorphism of underlying topological spaces of  $\mathcal{S}_l$  and  $Y_{k+l}$ .

We consider the following commutative diagram (5.26) obtained by the base change of (5.23) with a sequence of morphisms

$$\mathcal{S}_l \hookrightarrow B \times Y_{k+l} \times Y_k \rightarrow B \times Y_k \rightarrow Y_k.$$

Here  $B \times Y_{k+l} \times Y_k \rightarrow B \times Y_k$  is the natural projection:

$$(5.26) \quad \begin{array}{ccccccc} \mathcal{Z}_l^{\text{ns}} & \longrightarrow & B \times Y_{k+l} \times \mathcal{Z}^{\text{ns}} & \longrightarrow & B \times \mathcal{Z}^{\text{ns}} & \longrightarrow & \mathcal{Z}^{\text{ns}} \\ \downarrow u_0 & & \downarrow t_0 & & \downarrow s_0 & & \downarrow r_0 \\ \mathcal{Z}_l^\dagger & \longrightarrow & B \times Y_{k+l} \times \mathcal{Z}^\dagger & \longrightarrow & B \times \mathcal{Z}^\dagger & \longrightarrow & \mathcal{Z}^\dagger \\ \downarrow u_1 & & \downarrow t_1 & & \downarrow s_1 & & \downarrow r_1 \\ \mathcal{Z}_l & \xrightarrow{v'} & B \times Y_{k+l} \times \bar{\mathcal{Z}} & \longrightarrow & B \times \bar{\mathcal{Z}} & \longrightarrow & \bar{\mathcal{Z}} \\ \downarrow u_2 & & \downarrow t_2 & & \downarrow s_2 & & \downarrow r_2 \\ \cdot & \longrightarrow & B \times Y_{k+l} \times \hat{X}_k(f) & \longrightarrow & B \times \hat{X}_k(f) & \longrightarrow & \hat{X}_k(f) \\ \downarrow u_3 & & \downarrow t_3 & & \downarrow s_3 & & \downarrow \hat{\pi}_k \\ \mathcal{S}_l & \xrightarrow{v} & B \times Y_{k+l} \times Y_k & \longrightarrow & B \times Y_k & \longrightarrow & Y_k \end{array}$$

Let  $\mathcal{L}_l^\dagger$  be the line bundle on  $\mathcal{Z}_l^\dagger$  obtained by the pull back of  $L_1^\dagger$  by the morphisms in the above diagram (5.26). Let  $\mathcal{S}_{l,n}$  be the non-empty Zariski open subset of  $\mathcal{S}_l$  obtained by the inverse image of  $U_n$ . Since  $\dim H^0((B \times Z^\dagger)|_P, L_{1,P}^{\dagger \otimes n}) = G_n$  for  $P \in U_n$ , the direct image sheaf  $s_* L_1^{\dagger \otimes n}$  is a locally free sheaf of rank  $G_n$  on  $U_n$  and the natural map

$$s_* L_1^{\dagger \otimes n} \otimes \mathbf{C}(P) \rightarrow H^0((B \times Z^\dagger)|_P, L_{1,P}^{\dagger \otimes n})$$

is an isomorphism for  $P \in U_n$ . This follows by the Theorem of Grauert [H77] p.288, since  $U_n$  is reduced and irreducible. Here  $s : B \times Z^\dagger \rightarrow B \times Y_k$  is the natural map; i.e.,  $s = s_3 \circ s_2 \circ s_1$ . Let  $u$  be the morphism  $u : \mathcal{Z}_l^\dagger \rightarrow \mathcal{S}_l$  obtained by the composition  $u = u_3 \circ u_2 \circ u_1$ , where  $u_1, u_2, u_3$  are the morphisms in the above diagram (5.26). Then the natural map

$$u_* \mathcal{L}_l^{\dagger \otimes n} \otimes \mathbf{C}(P) \rightarrow H^0(\mathcal{Z}_l^\dagger|_P, \mathcal{L}_{l,P}^{\dagger \otimes n})$$

is also surjective, so an isomorphism on  $P \in \mathcal{S}_{l,n}$ . This follows by the Theorem of Cohomology and Base Change [H77] p. 290. Hence  $u_* \mathcal{L}_l^{\dagger \otimes n}$  is locally generated by  $G_n$  elements as an  $\mathcal{O}_{\mathcal{S}_l}$ -module on  $\mathcal{S}_{l,n} \subset \mathcal{S}_l$ . Let  $Y_{k+l,n} = q(\mathcal{S}_{l,n})$  be a non-empty Zariski open subset of  $Y_{k+l}$  (note that the under lying topological spaces of  $\mathcal{S}_l$  and  $Y_{k+l}$  are the same). Then by the above properties of  $q$ , the direct image sheaf  $(q \circ u)_* \mathcal{L}_l^{\dagger \otimes n}$  is locally generated by  $(l+1)G_n$  elements as a  $\mathcal{O}_{Y_{k+l}}$ -module on  $Y_{k+l,n} \cap Y_{k+l}^{\text{reg}}$ . Here, note that  $Y_{k+l}^{\text{reg}}$  is non-empty (otherwise  $f$  must be constant) and  $Y_{k+l}$  is irreducible. Hence  $Y_{k+l,n} \cap Y_{k+l}^{\text{reg}}$  is also non-empty.

Now look at the following commutative diagram

$$\begin{array}{ccccc} \mathcal{Z}_l^{\text{ns}} & & & & \\ \downarrow u_0 & & & & \\ \mathcal{Z}_l^\dagger & \xrightarrow{t_2 \circ v' \circ u_1} & B \times Y_{k+l} \times \hat{X}_k(f) & \xrightarrow{\psi} & \bar{B} \times Y_{k+l} \xrightarrow{\rho} \bar{B} \\ \downarrow q \circ u & & \downarrow \text{2nd proj} & & \downarrow \tau \\ Y_{k+l} & \xlongequal{\quad} & Y_{k+l} & \xlongequal{\quad} & Y_{k+l} \end{array}$$

where  $\rho$  is the first projection,  $\tau$  is the second projection and  $\psi$  is the morphism

$$\psi : B \times Y_{k+l} \times \hat{X}_k(f) \ni (a, v, w) \mapsto (a + \gamma_k(w), v) \in \bar{B} \times Y_{k+l}.$$

Since  $(\rho \circ \psi \circ t_2 \circ v' \circ u_1)^* L = \mathcal{L}_l^\dagger$ , we have a natural morphism

$$(5.27) \quad \tau_* \rho^* L^{\otimes n} = H^0(\bar{B}, L^{\otimes n}) \otimes_{\mathbf{C}} \mathcal{O}_{Y_{k+l}} \rightarrow (q \circ u)_* \mathcal{L}_l^{\dagger \otimes n}.$$

Here, note that  $\rho \circ \psi = \phi \circ \beta$  where  $\beta : B \times Y_{k+l} \times \hat{X}_k(f) \rightarrow B \times \hat{X}_k(f)$  is the morphism in the diagram (5.26) and  $\phi$  was defined by (5.24).

Now put  $I_n = \dim_{\mathbf{C}} H^0(\bar{B}, L^{\otimes n})$ . Then there is a positive integer  $n_0$  and positive constants  $C_1, C_2$  such that

$$I_n > C_1 n^{\dim \bar{B}}, \quad G_n < C_2 n^{\dim \bar{B}-2} \quad \text{for } n > n_0.$$

Here note that  $G_n = \dim_{\mathbf{C}} H^0(B \times Z^\dagger|_P, L_{1,P}^{\dagger \otimes n})$  for  $P \in \cap_{n \geq 1} U_n$ , and  $B \times Z^\dagger|_P = s^{-1}(P)$  has dimension  $\leq \dim \bar{B} - 2$ , for  $\text{codim}_{\hat{X}_k(f)} \bar{Z} \geq 2$  and  $\hat{\pi}_k \circ r_2 : \bar{Z} \rightarrow Y_k$  is dominant. Hence for a positive integer  $l$ , we can take a positive integer  $n(l)$  (e.g.  $\sim l^{3/4}$ ) such that

$$I_{n(l)} > (l+1)G_{n(l)}, \quad \lim_{l \rightarrow \infty} \frac{n(l)}{l} = 0.$$

Let  $\mathcal{F}$  be the kernel of (5.27) for  $n = n(l)$ ;

$$0 \rightarrow \mathcal{F} \rightarrow \tau_* \rho^* L^{\otimes n(l)} \rightarrow (q \circ u)_* \mathcal{L}_l^{\dagger \otimes n(l)} \quad (\text{exact}).$$

Then we have  $\mathcal{F} \neq 0$ . By taking the tensor of a sufficiently ample line bundle  $M_l$  on  $Y_{k+l}$  with  $\mathcal{F}$ , we may assume that  $H^0(Y_{k+l}, \mathcal{F} \otimes M_l) \neq 0$ . Since we have

$$\begin{aligned} H^0(Y_{k+l}, \mathcal{F} \otimes M_l) &\subset H^0(Y_{k+l}, (\tau_* \rho^* L^{\otimes n(l)}) \otimes M_l) \\ &= H^0(Y_{k+l}, \tau_*(\rho^* L^{\otimes n(l)} \otimes \tau^* M_l)) \\ &= H^0(\bar{B} \times Y_{k+l}, \rho^* L^{\otimes n(l)} \otimes \tau^* M_l), \end{aligned}$$

we may take a divisor  $F_l \subset \bar{B} \times Y_{k+l}$  which is defined by a non-zero global section of  $H^0(Y_{k+l}, \mathcal{F} \otimes M_l)$ . Then we have

$$\mathcal{Z}_l^{\text{ns}} \subset \psi^* F_l.$$

Here note that  $\mathcal{Z}_l^{\text{ns}} \subset \mathcal{Z}_l$  is an open immersion and  $\mathcal{Z}_l \xrightarrow{t_2 \circ v'} B \times Y_{k+l} \times \hat{X}_k(f)$  is a closed subscheme.

Using the decomposition  $A = B \times C$ , we let  $f_B : \mathbf{C} \rightarrow B$  be the holomorphic curve obtained by the composition of  $f$  and the first projection  $A \rightarrow B$ , and let  $f_C : \mathbf{C} \rightarrow C$  be the holomorphic curve obtained by the composition of  $f$  and the second projection  $A \rightarrow C$ . Now let  $a \in \mathbf{C}$  be a point such that  $J_k(f)(a) \in Z^{\text{ns}}$ . Put  $\tilde{f} : \mathbf{C} \rightarrow B \times Y_{k+l} \times \hat{X}_k(f)$  as

$$\tilde{f}(z) = (f_B(z) - f_B(a), \hat{\pi}_{k+l} \circ J_{k+l}(f)(z), J_k(f)(a)).$$

Then we have

$$\tilde{f}(\mathbf{C}) \subset B \times Y_{k+l} \times Z, \quad \tilde{f}(a) \in \text{Supp } \mathcal{Z}_l^{\text{ns}}, \quad \psi \circ \tilde{f} = J_{k+l}(f),$$

where the last equality holds under the identification  $\bar{B} \times Y_{k+l} = \hat{X}_{k+l}(f)$ .

Since  $v'$  is the base change of  $v$  in (5.26) and  $\tilde{f}$  factors through  $t_2$ , we have

$$\text{ord}_a \tilde{f}^* \mathcal{Z}_l = \text{ord}_a (t_3 \circ \tilde{f})^* \mathcal{S}_l,$$

hence by the construction of  $\mathcal{S}_l$  and Lemma 5.22, we have

$$\text{ord}_a \tilde{f}^* \mathcal{Z}_l = \text{ord}_a (J_{k+l}(f) - f(a))^* \mathcal{T}_{l, (J_k(f) - f(a))(a)}^\dagger \geq l + 1.$$

Hence we have

$$\text{ord}_a J_{k+l}(f)^* F_l = \text{ord}_a \tilde{f}^* \psi^* F_l \geq \text{ord}_a \tilde{f}^* \mathcal{Z}_l^{\text{ns}} = \text{ord}_a \tilde{f}^* \mathcal{Z}_l \geq l + 1.$$

Here note that we consider  $F_l$  as the divisor on  $\hat{X}_{k+l}(f)$  by the identification  $B \times Y_{k+l} \cong \hat{X}_{k+l}(f)$ , and  $\tau$  correspond to  $\pi_{k+l}$  by this identification. *Q.E.D.*

**(c) The end of the proof.** By (4.3) and (4.4) it suffices to show

$$N_{l_0}(r; J_k(f)^* Z) \leq \epsilon T_f(r) \|\epsilon, \quad \forall \epsilon > 0.$$

Note that

$$N_{l_0}(r; J_k(f)^* Z) \leq l_0 N_1(r; J_k(f)^* Z).$$

Furthermore, it suffices to prove

$$(5.28) \quad N_1(r; J_k(f)^* Z^{\text{ns}}) \leq \epsilon T_f(r) \|\epsilon, \quad \forall \epsilon > 0.$$

For we have

$$N_1(r; J_k(f)^* Z) = N_1(r; J_k(f)^* Z^{\text{ns}}) + N_1(r; J_k(f)^*(Z \setminus Z^{\text{ns}}))$$

and the second term of the right hand side is estimated to be at most “ $\epsilon T_f(r) \|\epsilon$ ” by induction on dimension of  $Z$ . Here note that  $\dim Z > \dim(Z \setminus Z^{\text{ns}})$ .

It follows from Lemma 5.21 and (5.15) that

$$\begin{aligned} (5.29) \quad (l+1)N_1(r; J_k(f)^* Z^{\text{ns}}) &\leq N(r; J_{k+l}(f)^* F_l) \leq T_{J_{k+l}(f)}(r; L(F_l)) \\ &= n(l)T_{\gamma_{k+l} \circ J_{k+l}(f)}(r; L) + T_{\pi_{k+l} \circ J_{k+l}(f)}(r; M_l) \\ &\leq n(l)T_{f_B}(r; L) + S_f(r). \end{aligned}$$

Using  $\lim_{l \rightarrow \infty} n(l)/(l+1) = 0$  and  $T_{f_B}(r; L) = O(T_f(r; D))$ , we obtain (5.28) and our Theorem 5.1.

## 6 Proof of Main Theorem

(a) Let the notation be as in the Main Theorem. The case of  $\text{codim}_{X_k(f)} Z \geq 2$  was finished by Theorem 5.1. Therefore we assume in the rest of this section that  $Z$  is a reduced Weil divisor  $D$  on  $A$ .

Set  $B = \text{St}(X_{k+1}(f))$ , which has a positive dimension (cf. (4.7)).

**Lemma 6.1** *Assume that  $D$  is irreducible and  $B \not\subset \text{St}(D)$ . Taking an embedding  $X_{k+1}(f) \hookrightarrow J_1(X_k(f))$ , we have*

$$\text{codim}_{X_{k+1}(f)}(X_{k+1}(f) \cap J_1(D)) \geq 2.$$

*Proof.* Let  $k = 0$ . It is first noted that  $J_1(A)$  is the holomorphic tangent bundle  $\mathbf{T}(A)$  over  $A$ , and  $X_1(f) \subset \mathbf{T}(A)$ .

Assume that  $\text{codim}_{X_1(f)}(X_1(f) \cap J_1(D)) = 1$ . Let  $E$  be an irreducible component of codimension 1 of  $X_1(f) \cap J_1(D)$ . Let  $\pi_1 : X_1(f) \rightarrow A$  be the natural projection. Then  $E$  is an irreducible component of  $X_1(f) \cap \pi_1^{-1}(D)$  and  $\overline{\pi_1(E)} = D$ .

Now  $\overline{\pi_1(E)} = D$  combined with  $B \not\subset \text{St}(D)$  implies that  $B$  can not stabilize  $E$ . Therefore  $B \cdot E$  (resp.  $B \cdot D$ ) contains an open subset of  $X_1(f)$  (resp.  $A$ ). In fact, since  $B$  and  $E$  are algebraic,  $B \cdot E$  contains a  $B$ -invariant Zariski open subset  $\Omega$  of  $X_1(f)$ .

Let  $p = f(z_0) \in f(\mathbf{C})$  be a point with the properties:

- (i) The orbit  $B \cdot p$  intersects  $D \setminus \text{Sing}(D)$  transversely in a point  $q$ ;
- (ii)  $J_1(f)(z_0) \in \Omega$ .

Then we choose an analytic 1-dimensional disk  $\Delta \subset B$  which contains the unit element  $e_B$  of  $B$  and we choose a non-empty open subset  $U$  of the non-singular part  $D^{\text{ns}}$  of  $D$  containing  $q$  such that

- (i) the map  $\phi : \Delta \times U \hookrightarrow A$  given by  $\phi(b, u) = b \cdot u$  is an open embedding,
- (ii) the subbundle  $\cup_{\zeta \in \Delta} \mathbf{T}(\{\zeta\} \times U) \subset \mathbf{T}(\Delta \times U)$  with  $\mathbf{T}(\{\zeta\} \times U) \cong \mathbf{T}(U)$  gives rise to a holomorphic foliation on  $\phi(\Delta \times U) \subset A$ .

Consider  $\hat{f}(z) = b \cdot f(z + z_0)$  with  $b \in B$  such that  $b \cdot p = q$  and  $p = f(z_0)$ . Note that  $\hat{f}(0) = b \cdot p = q$  and  $J_1(\hat{f})(0) = b \cdot J_1(f)(z_0) \in \Omega$ . Then there is an open neighbourhood  $W$  of 0 in  $\mathbf{C}$  such that  $J_1(\hat{f})(z) \in \Omega$  for all  $z \in W$ . Since  $\Omega \subset B \cdot E \subset B \cdot J_1(D)$ , it follows that  $\hat{f}'(z)$  is tangent to the leaves of the above defined foliation for all  $z \in W$ . By the identity principle this implies  $\hat{f}(\mathbf{C}) = b \cdot f(\mathbf{C}) \subset D$  which is absurd, since  $f$  is algebraically non-degenerate.

The proof for  $k \geq 1$  is similar to the above. *Q.E.D.*

(b) *Proof of the Main Theorem.* Let  $D = \sum_i D_i$  be the irreducible decomposition. By making use of Theorem 4.2 we have

$$\begin{aligned}
(6.2) \quad T(r; \omega_{\bar{D}, J_k(f)}) &\leq N_{k_0}(r; J_k(f)^* D) + S_f(r) \\
&\leq N_1(r; J_k(f)^* D) + k_0 \sum_{i < j} N_1(r; J_k(f)^*(D_i \cap D_j)) \\
&\quad + k_0 \sum_i N_1(r; J_{k+1}(f)^* J_1(D_i)) + S_f(r).
\end{aligned}$$

Since  $\text{codim}_{X_k(f)} D_i \cap D_j \geq 2$  for  $i \neq j$ , it follows from Theorem 5.1 that

$$k_0 \sum_{i < j} N_1(r; J_k(f)^*(D_i \cap D_j)) \leq \epsilon T_f(r) \|\epsilon, \quad \forall \epsilon > 0.$$

Note that  $J_{k+1}(f)^* J_1(D_i) = J_{k+1}(f)^*(X_{k+1}(f) \cap J_1(D_i))$ . If  $B \subset \text{St}(D_i)$ , then the image of  $D_i$  by  $X_k(f) \rightarrow X_k(f)/B$  is contained in a divisor on  $X_k(f)/B$ . Then as in (4.9) we infer that

$$N_1(r; J_{k+1}(f)^* J_1(D_i)) \leq N(r; J_k(f)^* D_i) \leq S_f(r).$$

Suppose that  $B \not\subset \text{St}(D_i)$ . It follows from Lemma 6.1 and Theorem 5.1 that

$$N_1(r; J_{k+1}(f)^* J_1(D)) \leq N_1(r; J_{k+1}(f)^*(X_{k+1}(f) \cap J_1(D_i))) \leq \epsilon T_f(r) \|\epsilon, \quad \forall \epsilon > 0.$$

Combining these with (6.2), we obtain

$$T(r; \omega_{\bar{D}, J_k(f)}) \leq N_1(r; f^* D) + \epsilon C T_f(r) \|\epsilon, \quad \forall \epsilon > 0,$$

where  $C$  is a positive constant independent of  $\epsilon$ . Now the proof of the Main Theorem is completed. *Q.E.D.*

## 7 Applications

(a) In [G74] M. Green discussed the algebraic degeneracy of a holomorphic curve  $f : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})$  omitting an effective reduced divisor  $D$  on  $\mathbf{P}^n(\mathbf{C})$  with normal crossings and of degree  $\geq n + 2$ . He proved the following theorem and conjectured that it would hold without the condition of finite order for  $f$ :

**Theorem 7.1** (M. Green [G74]) *Let  $f : \mathbf{C} \rightarrow \mathbf{P}^2(\mathbf{C})$  be a holomorphic curve of finite order and let  $[x_0, x_1, x_2]$  be the homogeneous coordinate system of  $\mathbf{P}^2(\mathbf{C})$ . Assume that  $f$  omits two lines  $\{x_i = 0\}, i = 1, 2$ , and the conic  $\{x_0^2 + x_1^2 + x_2^2 = 0\}$ . Then the image  $f(\mathbf{C})$  lies in a line or a conic.*

Here we answer his conjecture in more general form:

**Theorem 7.2** *Let  $f : \mathbf{C} \rightarrow \mathbf{P}^n(\mathbf{C})$  be a holomorphic curve and let  $[x_0, \dots, x_n]$  be the homogeneous coordinate system of  $\mathbf{P}^n(\mathbf{C})$ . Assume that  $f$  omits hyperplanes given by*

$$(7.3) \quad x_i = 0, \quad 1 \leq i \leq n,$$

*and a hypersurface defined by*

$$x_0^q + \dots + x_n^q = 0, \quad q \geq 2.$$

*Then  $f$  is algebraically degenerate.*

*Proof.* Let  $f(z) = [f_0(z), \dots, f_n(z)]$  be a reduced representation of  $f$ . Then  $f_i(z)$  have no zero for  $1 \leq i \leq n$ . The assumption implies the existence of an entire function  $h(z)$  such that

$$f_0^q(z) + \dots + f_n^q(z) = e^{h(z)}.$$

Write the above equation as

$$(f_0(z)e^{-h(z)/q})^q + \dots + (f_n(z)e^{-h(z)/q})^q = 1.$$

Changing the reduced representation of  $f$ , we may have that

$$(7.4) \quad f_1^q(z) + \dots + f_n^q(z) - 1 = -f_0^q(z).$$

Now we take a holomorphic curve into a semi-abelian variety  $A = (\mathbf{C}^*)^n$  with the natural coordinate system  $(x_1, \dots, x_n)$  defined by

$$g : z \in \mathbf{C} \rightarrow (f_1(z), \dots, f_n(z)) \in A.$$

Define a divisor  $D$  on  $A$  by

$$x_1^q + \dots + x_n^q - 1 = 0.$$

Let  $\bar{A}$  be a equivariant compactification in which  $D$  is generally positioned. Let  $\bar{D}$  be the closure of  $D$  in  $\bar{A}$ . Note that  $\text{St}(D) = \{0\}$  and that  $\text{ord}_z g^*D \geq 2$  for all  $z \in g^{-1}(D)$  by (7.4). Combining this with the Main Theorem ( $k = 0$ ), we see that for arbitrary  $\epsilon > 0$

$$\begin{aligned} T_g(r; L(\bar{D})) &\leq N_1(r; g^*D) + \epsilon T_g(r; L(\bar{D}))|_\epsilon \\ &\leq \frac{1}{q} N(r; g^*D) + \epsilon T_g(r; L(\bar{D}))|_\epsilon \\ &\leq \frac{1+q\epsilon}{q} T_g(r; L(\bar{D}))|_\epsilon. \end{aligned}$$

This leads to a contradiction for  $\epsilon < (q - 1)/q$ . *Q.E.D.*

*Remark.* The Zariski closure of the image  $f(\mathbf{C})$  can be more specified in terms of  $g$  defined in the above proof. It follows from [N98] that the Zariski closure of  $g(\mathbf{C})$  is a translate  $X$  of a proper semi-abelian subvariety of  $A$  such that  $X \cap D = \emptyset$ .

(b) Let  $A$  be a semi-abelian variety as above and let  $X \subset J_k(A)$  be an irreducible algebraic subvariety. We consider the existence problem of an algebraically nondegenerate entire holomorphic curve  $f : \mathbf{C} \rightarrow A$  such that  $J_k(f)(\mathbf{C}) \subset X$  and  $J_k(f)(\mathbf{C})$  is Zariski dense in  $X$ . This is a problem of a system of algebraic differential equations described by the equations defining the subvariety  $X$ .

The first necessary condition for the existence of such solution  $f$  is that  $\text{St}(X) \neq \{0\}$  (cf. (4.7)). Now we assume the existence of such  $f$ . Then we take a big line bundle  $L \rightarrow X$  and a section  $\sigma \in H^0(X, L)$  which defines a reduced divisor on  $X$ . We arbitrarily fix a trivialization

$$(7.5) \quad J_k(f)^*L \cong \mathbf{C} \times \mathbf{C},$$

and regard  $J_k(f)^*\sigma$  as an entire function.

**Theorem 7.6** *Let the notation be as above. Then there is no entire function  $\psi(z)$  such that every zero of  $\psi(z)$  has degree  $\geq 2$  and*

$$(7.7) \quad J_k(f)^*\sigma(z) = \psi(z), \quad z \in \mathbf{C}.$$

*In particular, there is no entire function  $\psi(z)$  satisfying*

$$(7.8) \quad J_k(f)^*\sigma(z) = (\psi(z))^q, \quad z \in \mathbf{C},$$

*where  $q \geq 2$  is an integer.*

*Remark.* The property given by (7.7) or (7.8) is independent of the choice of the trivialization (7.5).

*Proof.* Suppose that there is an entire function  $\psi(z)$  satisfying (7.7) or (7.8). Then it follows that

$$N_1(r; J_k(f)^*D) \leq \frac{1}{2}N(r; J_k(f)^*D).$$

Combining this with the Main Theorem, we infer the following contradiction:

$$T_{J_k(f)}(r; L) \leq \frac{1}{2}T_{J_k(f)}(r; L) + \epsilon T_{J_k(f)}(r; L) + o(r).$$

*Q.E.D.*

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